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Burn-in with Mixed Populations

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ABSTRACT

Any transistor (or other electronic component) from a production line may be either perfect or defective. The lifetime distributions of both types are assumed known. We focus here on the case where the perfect items never fail. Before any item is put in use, it is often the case that each production lot is tested to eliminate some of its defectives, i.e. the lot is subjected to burn-in. Here, the purpose of burn-in is to ensure with a given confidence level that an item chosen randomly from the test survivors has a given probability of operating properly for a given time period. This is the same as ensuring, after burn-in, the ratio of the number of defectives to the number of perfects is less than some bound with a desired level of probability.

Three without-replacement procedures are considered. Small sample theory is investigated for various assumptions about the information available concerning the number of defectives by using both analytic techniques and simulation. Large sample theory is studied, as well. It involves limiting distributions of order statistics, quantile processes and boundary crossing probabilities of a brownian bridge.

This study shows that the first two procedures are sensitive to the number of defective items assumed and the performance of the third procedure is not.

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19. ABSTRACT (Continue on reverse if necessary and identify by block number)

Any electronic component may be either perfect or defective. The lifetime distributions of both types are assumed known. We assume that perfect items never fail. Before being used, each production lot is tested to eliminate some of its defectives, i.e., the lot is subjected to burn-in. Here, the purpose of burn-in is to ensure with a given confidence level that an item chosen randomly from the test survivors has a given probability of operating properly for a given time period.

Three procedures are considered. Small sample theory is investigated for various assumptions about the information available concerning the number of defectives by using both analytic techniques and simulation. Large sample theory is studied, as well. This study shows that the first two procedures are sensitive to the number of defective items assumed and the performance of the third procedure is not.

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INTRODUCTION

§I.1 The Motivation

Suppose you plan to buy a personal computer (or any durable good) and there are several similar machines available on the market. Do you prefer to buy the one with free two-year service warranty or to buy the one without any warranty with the same price? Certainly, people prefer to buy the one with free two-year service warranty. How can the manufacturers offer this kind of two-year service free warranty without it costing them too much? Personal computers are mostly made of electronic components: microprocessor, TTL, ROM, RAM, PCB, etc.. In order to ensure a personal computer flawlessly working for two years, all of its components must be able to work perfectly for at least two years. Actually, some of its components are likely to fail during this two-year period. Therefore, the component replacement and the required service will cost the manufacturer some money. In order to prevent the occurrence of any failure during the warranty period and to reduce the replacement and service cost, the parts or components used to build this computer should have some very high pre-specified probability of surving for no less than two years. The underlined part of the above sentence is the (reliability) goal of this research. Here, three burn-in (burn-in will be defined in section I.3) procedures are designed to achieve this goal.

§ I.2 Defective Items and Good (Perfect) Items

Let's look at some useful characteristics of the electronic components, for example the semiconductors. (The application of this research is restricted to the burn-in test of semiconductors.) The interval right-after-manufacture during which a component (or item) may fail is often called as an early failure period or infant mortality period. Failures during this period are often referred as early failures or infant mortalities. The early failure can be caused by process defects or testing error or marginal design (Rickers (1978), or Jensen and Peterson (1982)). In this research, any item from a production line is classified as a defective item if its failure is caused by process defect or testing errors. The others are classified as good (or perfect) items. Like Jensen and Peterson (1982), we assume that the life time of any good item is considerably longer than that of any defective item i.e., the defective items deteriorate faster than the good items. So, the failure rate decreases as the testing time goes on.

As pointed out in many papers, the lifetime distributions of semiconductors from any production lot tend to have bimodal distributions: one mode for the subpopulation of the defective items and the other mode for the subpopulation of the good items. The percentage of the defective items is about 2 to 25 percent of the whole population. For example, the life tests on 1k RAMs and 16k ROMs, which is discussed in Edwards et al. (1978), show the proportions of the defective items from 2% to 5%; a study of transistor reliability at the Bell Laboratories (Peck and Zierdt (1974)) indicates 10% of defective items; and an experiment on CMOS transistors from RCA (Stitch et al. (1975)) shows 25% of defectives. Moreover, there are many more examples discussed in Jensen and Peterson (1982).

Several bimodal life time distributions of the good items and the defective items are proposed in the literature: mixtures of log-normal distribution by Holcomb and North (1984), by Jensen and Peterson (1982) and by Hallberg (1977); mixtures of Weibull distribution by Jensen and Peterson (1982 and 1979) and by Holcomb and North (1985).

The life time distributions of both the defective items and the perfect items are assumed known and exponential in this study. Several useful properties about the exponential distribution are used to develop the burn-in procedures designed in this research to achieve the (reliability) goal mentioned in §I.1.

If the life time distributions of both types of items are known but not exponential, each can be transformed into the exponential distributions by an appropriate probability (integral) transformation. However, one transformation usually will not make both distributions exponential, so this transformation can be useless to us. Fortunately, we can assume that the perfect items will never fail during burn-in and during the required service period, i.e., semiconductors exhibit infant mortality but not wear out. This kind of 'no wear out' property of semiconductors is pointed out in several papers: Holcomb and North (1985); Lawrence (1966); Blakemore, Kronson and von Alven (1963); Noris (1963); von Alven (1962); and von Alven and Blakemore (1961). Hence, in this study, the life time distributions of the defective items will be assumed known exponential with parameter (mean time to failure) 1 and the perfect items will be assumed to never fail, i.e., the mean time to failure of the perfect items is infinite.

§ I.3 Burn-In: the Approach to Achieve Reliability Goal

The cost to any manufacturer of giving its customers a free two-year service warranty can be minimized, if all the parts and components used to build his machines are all good items based on the previous section's discussion. However, no manufacturer can guarantee that any product from its production lines is always perfect. (As given in the previous section, the percentage of the defective items ranges from 2% to 25%.) Most manufactures admit that some portion of their products are not perfect. To reduce the number of defective items from a production line to a tolerable limit, we may try to improve the design of product, or use production process control, or do some after production inspection (Ricker 1978, or Jensen and Peterson 1982). In this research, only after production inspection is considered and three without replacement procedures are developed to eliminate some of the defective items and to reduce the number of defective items to an acceptable level. These three procedures require all the items of the production lot to be put on test (with stress) to remove the defective items through failure. This kind of test for electronic components is often called burn-in.

In this research, the purpose of burn-in is to achieve the reliability goal, given in § I.1, that is:

A very high probability, say α, is guaranteed srauch that any component which has survived burn-in will have a pre-specified chance, say ρ, to survive longer than its required service period. (I.3.1)

That is at most some fixed small proportion of the tested production lot could remain defective after the burn-in test is completed. Equation (I.3.1) will be formalized in the next section.

As pointed out before, the defective items tend to fail during their early lives, infant mortality. That is: the failure rate is decreased as burn-in goes on. So, burn-in can be used to eliminate the defective items effectively. Although, we don't want to put the defective items into the assembly line, we also don't want to waste our time and precious resources on any unnecessary burn-in. A very good discussion about the reasoning for considering burn-in is presented in Foster (1976). In addition, an excellent paper discussing all aspects of burn-in is Kuo and Kuo (1983).

§ I.4 Modeling

How can we be guaranteed that the desired reliability goal (I.3.1) is achieved after burn-in? For any production lot of semiconductors, the proportion of the defective items in this lot is unknown to us. We should try to reduce this proportion to an acceptable level. Here, burn-in is used to accomplish this. How can we say that this goal is obtained through burn-in? First, let's establish our mathematical model for the reliability goal (I.3.1).

First of all, we have the following notations:

m: the number of the defective items in a given burn-in lot when this lot is to be tested.

n: the size of burn-in lot or the total number of items being put on test.

t: the required service period (i.e., two years free service warranty period) of any items which passed burn-in.

T: duration of burn-in.

J_T: the number of failed defective items during a burn-in test with period T.

ρ: the pre-specified level of chance that any randomly chosen item which has passed burn-in would be able to have a useful service period t, i.e., the minimum required reliability for items which survive burn-in.

α: the desired minimum confidence level with which we assert that the reliability of those items which survive burn-in exceeds ρ.

F(s): the cumulative distribution function of the life time distribution of the defective items. Here, we assume $F(s) = 1 - \exp(-s)$ for s from 0 to ∞.

So, the reliability goal (I.3.1) can be formulated as

$$P(1-((m-J_T)/(n-J_T))(1-\exp(-t)) \geq \rho) \geq \alpha, \qquad (I.4.1)$$

where $m-J_T$ is the number of defective items left after burning-in is completed and which is still unknown after the test; $n-J_T$ is the number of items, defective and perfect items, which are still useful after burning-in; and $1-\exp(-t)$ is the chance that a passed defective item will fail before completing another t hours of service. Hence, $(m-J_T)/(n-J_T)$ is the conditional chance that any randomly chosen passed burn-in item is a defective one and $1 - ((m-J_T)/(n-J_T))(1-\exp(-t))$ is the chance that any item, which has passed a burn-in test with duration T, can complete a service with period t, i.e., the reliability of a randomly chosen item which has survived burn-in is

$$R(t; T, m, n) = 1 - ((m-J_T)/(n-J_T))(1-\exp(-t)). \qquad (I.4.2)$$

Notes:

1. Here, J_T is random and T can be a fixed value or given by the stopping rule used.
2. $R(t; T, m, n) = 1 - ((m-J_T)/(n-J_T))(1-\exp(-t))$ (I.4.3)

   $R(t; T, m, n) \geq \rho$

   $\Leftrightarrow J_T \geq (m(1-\exp(-t))-n(1-r))/(r -\exp(-t))$

   $\Leftrightarrow J_T \geq m_0,$ (I.4.4)

   where

   $m_0 = [m_0^*]$ if $m_0^* = [m_0^*]$ or $m_0 = [m_0^*]+1$ if $m_0^* > [m_0^*]$ and

   $m_0^* = (m(1-\exp(-t))-n(1-\rho))/(\rho -\exp(-t)).$ (I.4.5)

   (Note: Define [x] as the greatest integer less than or equal to x.)
3. $P(1 - ((m-J_T)/(n-J_T))(1-\exp(-t)) \geq \rho) \geq \alpha$

   $\Leftrightarrow P(J_T \geq m_0) \geq \alpha.$ (I.4.6)

Therefore, the reliability goal (I.4.1) is achieved if the probability of eliminating at least m_0 defective items through burn-in is at least α.

4. The number of the defective items, m, in a burn-in lot is unknown, so m_0 is unknown. A reasonable value of m or its estimator can be used to develop the burn-in procedures.

5. Both m_0^* and m_0 are non-decreasing functions of m. If the assumed value of m is larger than its true value, then a conservative rule is obtained.

6. If $t = \infty$, $m_0^* = (m-n(1-r))/\rho$. This is the case that the desired service period of any passed burn-in item is infinite.

7. Rewriting (I.4.4), we have

   $J_T \geq m_0$

   $$\Longleftrightarrow m - J_T \leq m - m_0, \qquad \text{(I.4.7)}$$

   where $m - J_T$ is the number of the defective items passing burn-in. So, the reliability goal (I.4.1) can be obtained, if a conservative upper bound, say μ, for $m - J_T$ is given. That is to find a stopping rule such that

   $$P(m - J_T \leq \mu) \geq \alpha. \qquad \text{(I.4.8)}$$

   Marcus and Blumenthal (1974) develop a very good screening procedure to ensure (I.4.8). Their idea will be investigated further in chapter 2.

§ I.5 Existing Non-Replacement Burn-In Procedures

There are many burn-in procedures existing in the literature. For any item which failed during burn-in, some of these procedures do not replace this item with an untested item, while some of them do. The three burn-in procedures developed in this research are all non-replacement procedures. Hence, let's only look at the existing non-replacement burn-in procedures. Since the with-replacement procedures are not our focus, they will not be discussed here. From now on, the burn-in procedures will be the procedures without replacement.

Basically, the burn-in procedures can be classified into the following two categories: (1) Sequential Procedure: Marcus and Blumenthal (1974); (2) Fixed Time Procedure: Lawrence (1966), Washburn (1970), and Watson and Wells (1961).

Let's look at the results that these procedures obtained:

(a) Marcus and Blumenthal (1974): A sequential screening procedure is obtained such that the remaining number of defective items is less than some pre-specified number with, at least, a desired probability.

(b) Lawrence (1966): Sharp upper and lower bounds on the burn-in time to achieve a desired mean residual life are obtained.

(c) Washburn (1970): A mathematical model is established based on cost considerations. Moreover, the optimal burn-in time is derived to achieve the maximum performance of this model under total cost constraint.

(d) Watson and Wells (1961): the lower bound of the probability that the mean remaining life is greater than some specified lower bound is obtained.

From the above discussion, we know that, except for Marcus and Blumenthal (1974), none of the above procedures can be used to achieve our reliability goal (I.4.1). In order to achieve (I.4.1), three burn-in procedures are proposed here as mentioned before. Among these three procedures, one of them is based on Marcus and Blumenthal(1974). If we compare the inequality (I.4.7) in the previous section and the inequality (2.1) of their paper, we see that they are the same. This is a very good starting point.

§ I.6 The Ideas of the Three Procedures Developed in This Research

If m, the number of the defective items, is known, then there is no difficulty for us to obtain (I.4.1) by never stopping burn-in until the m_0-th failed defective item is observed. However, m is unknown. An assumed value of m can be used to design a burn-in procedure to achieve (I.4.1): Procedure 0 and Procedure I are derived through this approach. Alternatively, a statistical estimator of m can also be used to design such a burn-in procedure: Procedure II is developed through this approach.

If an assumed value of m is used, we can find a burn-in length, say ∂ such that the probability that the life time of the m_0-th failed defective item are less than ∂ is at least α and ∂ is the duration of burn-in. This is the idea of Procedure 0. Or, we can find a value, say t^* and calculate the waiting times between successive failures of the defective items such that the probability of the first m_0 waiting times are all less than t^* is at least α. Burn-in continues until some waiting time exceeds t^*. Hence, the chance of eliminating at least m_0 defective items through burn-in is at least α. This the idea of Procedure I. Why does this work for Procedure I ? The reason is that the waiting time between the ith and i+1st failure of the defective items is stochastically less than the waiting time between the i+1st and the i+2nd failure of the defective items.

If the maximum likelihood estimator of m, $m^{est} = J_T/(1-\exp(-T))$ (Johnson 1961), is used to replace m in (I.4.2), We have

$$1- ((m^{est}-J_T)/(n-J_T))(1-\exp(-t)) \geq \rho \qquad (I.6.1)$$

$$<=> 1 - J_T(1/(1-\exp(-T) - 1)(1-\exp(-t))/(n-J_T) \geq \rho$$

$$<=> T \geq \ln\{[J_T/(n-J_T)][(1-\exp(-T))/(1-k)] + 1\}. \qquad (I.6.2)$$

A sequence of stopping times can be obtained by using (I.6.2). This is the idea of Procedure II.

CHAPTER I

PROCEDURE 0

§1.1 Introduction

Any electronic component from a production lot can be a good one or a defective one. The lifetime distribution of any normal one is assumed known and longer than its useful period. The life time of any defective one is assumed continuous and to have the same distribution as the other defectives. Here, we assume that the lifetime distribution of the defectives in any burn-in lot is known, and that the lifetimes are independent. In this case, we can assume that they are independent exponential random variables with parameter λ equal to 1.

When a randomly chosen item is selected from a production line, we don't know if it can survive for a given time period. In order to ensure that this item has good performance with a desired level of probability, it is often the case that each production lot is put on burn-in to eliminate some of its defectives.

A brief summary of this chapter is the following: Section Two is the basic idea for this stopping rule and the stopping time ∂. Some of the properties of this approach will be used in the later chapters. A traditional large sample approach to obtain ∂ is given in Section Three. The relation between ∂ and several important parameters, like the size of the defective items m, is discussed in Section Four. A numerical computation algorithm for ∂ is presented in Section Five and a theorem to show how this algorithm works is given in Section Six. Under the condition that m/n is a constant, the relation between ∂ and m is discussed in Section Seven. Additional relations between ∂ and m is considered in Section Eight, too. The number of the defective items which may left after this screening procedure is stopped is consider in Section Nine. In the last section, the expected reliability is computed if this stopping rule is used.

§1.2 Idea and the Stopping Rule 0

For a given burn-in lot, assume m is the total number of defective items in a burn-in lot of size n. Our goal is to find the stopping rules which can ensure that the chance is at least ρ that any randomly picked item from this lot will survive for a period t after a burn-in with period D and our confidence in this chance is at least α. Formulating this, we have

$P(R(t; D, m, n) \geq \rho) \geq \alpha$

where $R(t; D, m, n)=1-\{(m-J_D)/(n-J_D)\}\bullet P(T \geq D+t \mid T \geq D)$, D is the duration of burn-in, and J_D is the number of defectives that failed during burn-in up to time D.

To ensure $P(R(t; D, m, n) \geq \rho) \geq \alpha$, the following lemma tells us the number of defectives in the burn-in lot which must be eliminated through burn-in when all the information about this lot is available.

Lemma 1.2.1

For $0 < \exp(-t) < \rho < 1$, $t > 0$, $D > 0$ and $0 < m < n$,

$$R(t; D, m, n) \geq \rho) \tag{1.2.1}$$

can be ensured by screening out at least m_0 defectives from this burn-in lot,

$$m_0^* = \{m(1-\exp(-t)) - n(1-\rho)\}/(\rho-\exp(-t)) \text{ and} \tag{1.2.2}$$

$$m_0 = \text{the smallest integer greater than or equal } m_0^*. \tag{1.2.3}$$

Proof:

$R(t; D, m, n) = 1 - [(m-J_D)/(n-J_D)]\bullet P(T \leq t+D \mid T \geq D)$

$= 1- [(m-J_D)/(n-J_D)]\bullet[1-\exp(-t)]$.

So, $R(t; D, m, n) \geq \rho$

$\iff (m-J_D)\cdot(1-\exp(-t)) \le (1-\rho)\cdot(n-J_D)$

$\iff m\cdot(1-\exp(-t)) - n\cdot(1-\rho) \le J_D\cdot(\rho-\exp(-t))$ (1.2.4)

$\iff J_D \ge \{m(1-\exp(-t)) - n(1-\rho)\}/(\rho-\exp(-t)) = mo^*$, since $\rho > \exp(-t)$.

$\iff J_D \ge mo$.

The proof of this lemma is completed.

Using this lemma, we have the following very useful lemma.

Lemma 1.2.2

For $\rho > 0$, $\alpha > 0$, $t > 0$ and integers $n \ge m \ge 0$, to obtain our reliability goal $R(t; D, m, n) \ge \rho$, burn-in is required and useful if and only if

$(n-1) \ge m \ge 1 + (n-1)\cdot(1-\rho)/(1-\exp(-t))$. (1.2.5)

Note:

The inequality (1.2.5) tells us if burn-in is useful then

$1 > m/n > (1-\rho)/(1-\exp(-t))$ if $\rho > \exp(-t))$. (1.2.6)

So, when $\rho > \exp(-t))$, no burn-in is required if $m/n < (1-\rho)/(1-\exp(-t))$. From now on, to simplify our calculation, we will say that

burn-in is useful and required if (1.2.6) is satisfied. In addition, the stopping rule obtained under (1.2.6) is more conservative than the stopping rule under (1.2.5).

Proof:

If m=n then burn-in will not be able to improve the reliability, since the only items left after burn-in are always defective. So, burn-in is required only when $m \le n-1$. In addition, from (1.2.4) of the above lemma, we know that burn-in is needed and our reliability goal is achieved if and only if, for some for some J_D with $m \ge J_D \ge 1$,

$m\cdot(1-\exp(-t)) - n\cdot(1-\rho) \ge J_D\cdot(\rho-\exp(-t))$. (1.2.7)

$\iff m\cdot(1-\exp(-t)) \ge (n - J_D)\cdot(1-\rho) + J_D\cdot(1-\exp(-t))$

for some J_D with $m \geq J_D \geq 1$.

$\Leftrightarrow m \cdot (1-\exp(-t)) \geq (n-1) \cdot (1-\rho) + 1 \cdot (1-\exp(-t))$. this is the case that $J_D = 1$.

$\Leftrightarrow m \geq 1 + (n-1) \cdot (1-\rho)/(1-\exp(-t))$. The proof of this theorem is completed.

Define

$r = m/n$

$s_{m0^*} = m0^*/m = \{[1-\exp(-t)] - (1/r) \cdot (1-\rho)\}/(\rho-\exp(-t))$,

$s_{m0} = m0/m$ and

$s_{(n-1)0} = (n-1)0/(n-1)$ where $(n-1)0$ is $m0$ when $m = n-1$.

Note:

It's trivial that $0 \leq r < 1$.

When $\rho > \exp(-t)$ and $0 \leq r < 1$, $1 > s_{m0^*} > 0$, if and only if

$$1 > r > (1-\rho)/(1-\exp(-t)). \tag{1.2.8}$$

This is the same as (1.2.6), the condition for burn-in.

We have $s_{m0} \sim s_{m0^*}$ (where "A ~ B" means that A is approximately equal to B) and $s_{(n-1)0}$ is the largest possible s_{m0}. This ratio s_{m0} is always greater than 0 when the production lot is required to be tested. In addition, $s_{m0} < s_{(n-1)0} < 1$ (and $s_{(n-1)0} - s_{m0^*} = (1/r-1) \cdot (1-\rho)/(\rho-\exp(-t)) > 0$), so the burn-in procedure using a stopping rule based on this idea will terminate with probability one and (1.2.1) will be achieved. In addition, $s_{m0} \sim s_{m0^*}$ is an increasing function in r which is an unknown constant and depends on m where m is unknown.We'd like to use some estimate of m, say the upper (lower) bound of m. If r is more (less) than the true r, then $m0/m$ or $m0$ will be more (less) than its true value. Hence, in the true case, $P(R(t; D, m, n) \geq \rho)$ is always more (less) than α if the stopping rule is chosen so that $P(R(t; D, m, n) \geq \rho) \geq \alpha$ is true with m replaced by its upper (lower) bound. So, we have the following lemma.

Lemma 1.2.3:

If the assumed number of defectives, m, is more (less) than its true value in a burn-in lot then a larger (smaller) portion of defective items than necessary will be eliminated through burn-in. In addition, the duration of burn-in will be longer (shorter) than is truly needed.

Let's use the above result to define the **Stopping Rule 0.0**:

Stop burn-in when the total number of the observed failed defectives reaches m_0. (S.1.1)

As described before, m is not clearly known in the real situation. This rule needs some modification. (Note: Stopping rule (S.1.1) assures $P(R;t,D,m) \geq \rho) = 1$.) Sometimes an upper bound or(and) a lower bound of m is available. Sometimes the prior distribution of m is known. Suppose this stopping rule is used with m replaced by an estimate of the upper bound then we might wait forever before the screening procedure is stopped simply because the number mo used is larger than the true value of m (i.e. the number of defectives in this production lot is over-estimated).

For the case of interest to us, the lifetime distributions of defectives are independent, identical and known. Our reliability goal, to screen out at least mo defectives with acceptable high probability, can be achieved by never stopping burn-in before some fixed duration of burn-in, say ∂. Let D be the duration of burn-in. We have the following revised **Stopping Rule 0.1**:

Stop burn-in at the First Time When $D \geq \partial$, (S.1.2)

where ∂ is the lower bound of the duration of burn-in which will ensure $P(J_D \geq m_0 | D \geq \partial) \geq \alpha$.

§1.3 The Determination of ∂: a Large Sample Theory Approximation

How is ∂ determined? This is the topic of this section and §1.5. To screen out $s_{m0}= m0/m$ proportion of defectives from burn-in, we may try to use $\partial = F^{-1}(m0/m)$ where F is the cumulative distribution function of the failure time of the defective items. But this is often not the right choice for us to guarantee that we will have $P(R(t; D, m, n) \geq \rho) \geq \alpha$ when this stopping rule is used. As we may assume that an upper bound of m is the true m in the previous section, we'll assume that we know m in deriving ∂. More discussions about the relation between Δ ($=F(\partial)$) and m (and other parameters) will be given in the next section.

Let T_i be the failure time of the ith failed defective and $\Delta = 1-\exp(-\partial)$. We have the following equivalent inequalities:

$$P(R(t; D=\partial, m, n) \geq \rho) \geq \alpha \tag{1.3.1}$$

$$\Longleftrightarrow P(J_\partial \geq m0) \geq \alpha$$

$$\Longleftrightarrow P(T_{m0} \leq \partial) \geq \alpha \tag{1.3.2}$$

$$\Longleftrightarrow P(U_{m0} \leq \Delta) \geq \alpha \tag{1.3.3}$$

where $U_{m0} = 1 - \exp(-T_{m0})$. Hence, to ensure (1.3.1), we need to find a ∂ in (1.3.2) or a Δ in (1.3.3) which will make these inequalities hold.

We have

$$P(U_{m0} \leq \Delta) = \int_{0<u<\Delta} [m!/[(m0 -1)!(m-m0)!]]\, u^{(m0-1)}\cdot(1-u)^{(m-m0)}\, du$$

$$= \Sigma_{i=m0,m}\{m!/[(m-i)!i!)]\}\Delta^i (1-\Delta)^{m-i} \tag{1.3.4}$$

This is an incomplete beta integral, or a partial binomial sum. For a given α, we need the Δ with $P(U_{m0} \leq \Delta) = \alpha$. This can be found by using the existing tables of the binomial distributions (when m is of small or moderate size). A binomial distribution table may not be at hand or it may not cover all the values of our interest. Can we do something other than this? Traditionally, we use a normal approximation and large sample theory to approximate Δ. In this section, we'll consider this approach first. In addition to this, in §1.5, a direct computation scheme to calculate Δ will be developed.

Let's try to obtain an approximated Δ for

$$P(U_{m0} < \Delta) = \alpha \tag{1.3.5}$$

by using the large sample theory of order statistics (e.g. Cramer 1945 or Smirnov 1962). We have

$$\sqrt{m} \bullet \{ (U_{m0} - s_{m0}) / \sqrt{[s_{m0}(1-s_{m0})]} \} \longrightarrow N(0,1), \text{ if } 0< s_{m0}<1, \tag{1.3.6}$$

where $s_{m0} = m0/m$.

Using (1.3.6), Δ can be approximated easily as the following, where z_α is the $100 \bullet \alpha$ percentile from $N(0,1)$. Letting

$$\sqrt{m} \bullet \{ (\Delta - s_{m0}) / \sqrt{[s_{m0}(1-s_{m0})]} \} = z_\alpha, \tag{1.3.7}$$

and solving it for Δ, we have

$$\Delta = s_{m0} + z_\alpha \sqrt{[s_{m0}(1-s_{m0})]} / \sqrt{m}. \tag{1.3.8}$$

In addition, assuming $s_{m0} \longrightarrow s_0$ as $m \longrightarrow \infty$, we have

$$s_0 = [1-\exp(-t)]/[\rho-\exp(-t)]-(1/r) \bullet [(1-\rho) /(\rho-\exp(-t)] = s_{m0*}.$$

Using Slutsky's Theorem,

$$\sqrt{m} \bullet \{ (U_{m0} - s_0) / \sqrt{[s_0(1-s_0)]} \} \dashrightarrow N(0,1), \text{ if } 0<s_0<1. \tag{1.3.9}$$

Similarly, we have

$$\Delta = s_0 + z_\alpha \sqrt{[s_0(1-s_0)]} / \sqrt{m}. \tag{1.3.10}$$

So, ∂ can be, by using (1.3.8),

$$\partial = -\ln(1 - s_{m0} - z_{\alpha} \sqrt{[s_{m0}(1-s_{m0})]} / \sqrt{m}) \tag{1.3.11}$$

or, by using (1.3.10),

$$\partial = -\ln(1 - s_0 - z_{\alpha} \sqrt{[s_0(1-s_0)]} / \sqrt{m}) \tag{1.3.12}$$

Hence, we have the following theorem.

Theorem 1.3.1

When the assumed m is large enough (≥25) and r=m/n, the fixed burn-in duration of this stopping rule, (S.1.1), is equation (1.3.11) or (1.3.12).

§1.4.1 How ρ and m Affect Δ (or ∂)

We can see if m is overestimated then the duration of burn-in could be much longer than what is truly required. In the previous section, Δ is expressed as the sum of two terms: s_{m0} (or s_0) and $z_\alpha\sqrt{[s_{m0}(1-s_{m0})]} / \sqrt{m}$ (or $z_\alpha\sqrt{[s_0(1-s_0)]} / \sqrt{m}$).

For s_{m0} (or s_0), we have defined

$s_0 = s_{m0^*} = \{r(1-\exp(-t))-(1-\rho)\}/\{r(\rho-\exp(-t))\}$

$= \{(1-\exp(-t))/(\rho-\exp(-t))\} - \{(1-\rho)/[r(\rho-\exp(-t))]\}$ (1.4.1)

By Lemma 1.2.3, s_{m0^*} (or s_0) is a monotonically increasing function of r. In addition,

$s_{m0^*} = s_0 = (1/r) - [(1-\exp(-t))(1-r)]/[r(\rho-\exp(-t)]$ (1.4.2)

is a monotonically increasing function of ρ, too.

For the second term, $z_\alpha\sqrt{[s_{m0}1-s_{m0})]} / \sqrt{m}$ (or $z_\alpha\sqrt{[s_0(1-s_0)]} / \sqrt{m}$) , its value is mainly determined by $\sqrt{m}$. This term is not very significant in the determination of Δ if m is large enough.

Hence, the value of Δ is mainly determined by s_{m0} (or s_0) if m is large enough. The two figures in the next page can give us the idea about the relation between s_{m0} (or s_0) and Δ, and the relation between s_{m0} (or s_0) and ∂ when m is sufficiently large, and ρ and t are fixed.

From the two figures in the next page, we can see that the duration of burn-in could be extremely long when s_{m0} (or s_0) is quite close to 1. This tells us that when the s_{m0} (or s_0) used is over-estimated we should be very careful, otherwise we will waste a large amount of time in extra burn-in. On the other hand, if s_{m0} (or s_0) is under-estimated, the reliability goal $P(R(t; D, m, n) \geq \rho) \geq \alpha$ may not be achieved. In reliability context, ρ is usually very close to 1. This will force s_{m0} (or s_0) close to 1, too. Hence, we should be very careful when this (fixed time) stopping rule is used.

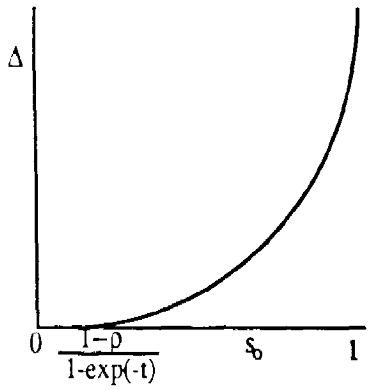

Figure 1

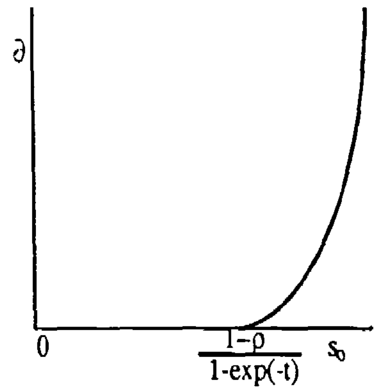

Figure 2

§1.5 The Determination of ∂: a Direct Numerical Calculation

Those ∂'s and Δ's derived in §1.3 are not appropriate for us to use if any one of the following occur: n, the lot size, is too small; m, the number of the defective items in the burn-in lot, is bounded above by a small number; or an accurate value of ∂ (or Δ) is required. We should use the Δ by solving (1.3.5) directly if no suitable table of binomial distributions is available, i.e., to solve equation (1.3.5), $P(U_{m0} \leq \Delta) = \alpha$, numerically.

If we define

$$h(\Delta) = P(U_{m0} < \Delta) = \Sigma_{i=m0,m}\{m!/[(m-i)!i!)]\}\Delta^i (1-\Delta)^{m-i}. \qquad (1.5.1)$$

The following lemma is clear to us.

Lemma 1.5.1:

For Δ in (0,1), $h(\Delta)$ is a differentiable function with a positive and bounded derivative in Δ, i.e., $h(\Delta)$ is bounded, continuous and monotonically increasing in Δ. Note that $h(\Delta)$ is a one-one function from [0,1] to [0,1].

Proof:

We can consider $h(\Delta)$ as the c.d.f. of a uniform order statistic.

$h'(\Delta)$ is a p.d.f. of a beta distribution with parameters $\alpha=m0$ and $\beta=m-m0+1$ being positive integers. So, the proof of this lemma is trivial.

Hence, we have the following binary search to find the unique Δ with $h(\Delta)=\alpha$.

A binary search to find the solution of $h(\Delta)=\alpha$.

1) $\Delta_0=0.5$.

2) $\Delta_{i+1}=\Delta_i+(0.5)^{i+1}$ if $h(\Delta_i) < \alpha$.

$\Delta_{i+1}=\Delta_i-(0.5)^{i+1}$ if $h(\Delta_i) > \alpha$.

$\Delta_{i+1}=\Delta_i$ if $h(\Delta_i) = \alpha$.

3) Stop if $|\Delta_{i+1}-\Delta_i|\le e$ where e is the given error bound.

After 30 iterations, we shall have $|\Delta_{i+1}-\Delta|<10^{-9}$, the Δ value derived through this algorithm which is close to its true value with error less than 10^{-9}. We'll obtain the corresponding ∂ by letting $\partial=-\ln(1-\exp(-\Delta))$. The following theorem and its corollary show that the Δ derived through this binary search converges to the solution of (1.5.1) = α.

Theorem 1.5.2:

If $h(\Delta)$ is a differentiable function with a bounded positive derivative for Δ in (0,1), then the search algorithm defined in above converges to the Δ with $h(\Delta) = \alpha$.

Proof:

For any two positive integer j > i > 0, we have

$$0 < |\Delta_j - \Delta_i| < (1/2)^i. \qquad (1.5.2)$$

In addition,

$0 \le \text{limit}_{i \to \infty} |\Delta_j - \Delta_i| \le \text{limit}_{i \to \infty} (1/2)^i = 0.$

Hence, $\{\Delta_i\}_{i=1,\infty}$ is a convergent Cauchy sequence which implies that it's a convergent sequence. So, let $\text{limit}_{i \to \infty} \Delta_i = \Delta^*$.

Assume, for Δ in (0,1), $|h'(\Delta)| \le \Omega < \infty$.

$\text{limit}_{i \to \infty} |h(\Delta_j) - h(\Delta_i)| \le \text{limit}_{i \to \infty} \max_{\Delta i \le \Delta \le \Delta j} |h'(\Delta)| \cdot |\Delta_j - \Delta_i|$

$< \Omega \cdot \text{limit}_{i \to \infty} |\Delta_j - \Delta_i| = 0$. So, $\{h(\Delta_i)\}$ is convergent.

The last step is to show that $\text{limit}_{i \to \infty} h(\Delta_i) = h(\Delta^*) = \alpha$.

Since $h(\Delta)$ is continuous, so $\text{limit}_{i \to \infty} h(\Delta_i) = h(\Delta^*)$.

Suppose $h(\Delta^*) \neq \alpha$. Since $h(\Delta)$ is a one to one mapping from [0,1] onto [0,1] with bounded first derivative, we know that $h(\Delta^*) \neq \alpha$ cannot happen.

The proof of this theorem is completed.

So, this binary search algorithm can be used to derive the desired Δ up to any desired precision. Similar binary search algorithms will be used in the other sequential screen procedures studied in the following chapters.

§1.6 The Δ (or ∂), from the direct numerical calculation, is an increasing function in m, α and ρ when n Is Fixed.

Define $\Delta(m,\alpha,\rho)$ as the solution of (1.5.1), and $h(\Delta) = P(U_{m0} < \Delta) = \Sigma_{i=m0,m}\{m!/[(m-i)!i!)]\}\Delta^i (1-\Delta)^{m-i} = \alpha$, where m0 = the least integer greater than or equal $\{m(1-\exp(-t)) - n(1-\rho)\}/(\rho-\exp(-t))$. We have m0 (or m0*) is an increasing function in ρ. Using (1.5.1) and using the property of order statistics, we have the following theorem (without proof).

Theorem 1.6.1:

$\Delta(m,\alpha,\rho)$ is an increasing function in ρ.

From lemma 1.5.1, we known that $h(\Delta)$ is an increasing function in Δ. Hence, we have the following trivial result.

Theorem 1.6.2:

$\Delta(m,\alpha,\rho)$ is an increasing function in α.

Before we study the relation between $\Delta(m,\alpha,\rho)$ and m, let m1 and m2 be two positive integers with m1 < m2, a1 and a2 be two positive real numbers with a1 ≤ m1 and a2 ≤ m2, and a1o and a2o be the smallest integers larger than or equal to a1 and a2, respectively. We have the following lemma.

Lemma 1.6.1:

(i) If m1 - a1 ≥ m2 - a2, then m1 - a1o ≥ m2 - a2o.

(ii) If m1 - a1 ≤ m2 - a2, then m1 - a1o ≤ m2 - a2o.

Proof:

(i) 1. If there is an integer i such that

$m1 - a1 \geq i > m2 - a2$,

then we have

$m1 - a1 \geq m1 - a1_0 \geq i > m2\text{-}a2 \geq m2 - a2_0$.

2. If there is an integer i such that

$i > m1 - a1 > m2 - a2 \geq i - 1$.

then

$i > m1 - a1 > m2 - a2 \geq m1 - a1_0 = m2 - a2_0 = i - 1$.

3. If $m1 - a1 = m2 - a2$, then $m1 - a1_0 = m2 - a2_0$.

1., 2. and 3. prove that (i) is true.

Similarly, we have the following proof for ii).

(ii) 4. If there is an integer i such that

$m1 - a1 < i \leq m2 - a2$,

then we have

$m2\text{-}a2 \geq m2 - a2_0 \geq i > m1 - a1 \geq m1 - a1_0$.

5. If there is an integer i such that

$i > m2 - a2 > m1 - a1 \geq i - 1$.

then

$i > m2 - a2 > m1 - a1 \geq m2 - a2_0 = m1 - a1_0 = i - 1$.

6. If $m1 - a1 = m2 - a2$, then $m1 - a1_0 = m2 - a2_0$.

4., 5. and 6. prove that (ii) is true.

The proof of this lemma is completed.

Let m1 and m2 be two positive integers, m1 < m2 defined as before Lemma 1.6.1, with values less than two integers n1 and n2, respectively, and

$mio^* = [mi \cdot (1-\exp(-t))-ni \cdot (1-\rho)] / [\rho-\exp(-t)]$

mio = the least integer greater than or equal $[mi \cdot (1-\exp(-t))-ni \cdot (1-\rho)] / [\rho-\exp(-t)]$

where i=1 or 2.

The following corollary of the above Lemma 1.6.1 is very useful in comparing the duration of burn-in for different production lots in the same or different burn-in facilities (same n or different n), or in determining the appropriate burn-in lot size under the time and cost constraints. Applications of Lemma 1.6.1 and Corollary 1.6.1 will be seen in this procedure and the other procedures.

Corollary 1.6.1:

1) If n2 = n1 and m2 > m1, then

$m2o^* > m1o^*$, $m2o \geq m1o$, $m1 - m1o^* > m2 - m2o^*$ and $m1 - m1o \geq m2 - m2o$.

2) If n2 > n1 m2 > m1, $m1/n1 = m2/n2 = r$ and $r > (1-\rho)/(1-\exp(-t))$, then $m2o^* > m1o^*$, $m2o \geq m1o$, $m1 - m1o^* < m2 - m2o^*$ and $m1 - m1o \leq m2 - m2o$.

Note: Using Lemma 1.2.2, $r > (1-\rho)/(1-\exp(-t))$ means that burn-in is required for this production lot.

Proof:

Let, for i = 1 and 2, $ai = mio^*$ and $aio = mio$.

For i=1,2

$mi-mio^* = mi-[mi \cdot (1-\exp(-t))-ni \cdot (1-\rho)] / [\rho-\exp(-t)]$.

1) We only need to prove that $m1 - m1o^* > m2 - m2o^*$, the other results are trivial,

$(m1-m1o^*) - (m2-m2o^*)$, since n1=n2,

$= \{-m1 \cdot [(1-\rho)/(\rho-\exp(-t))]\}-\{-m2 \cdot [(1-\rho)/(\rho-\exp(-t))]\}$

$= (m2-m1)\cdot[(1-\rho)/(\rho-\exp(-t))] > 0.$

2) Similar to 1), we need to prove m1 - m1o* < m2 - m2o*.

$(m1-m1o^*) - (m2-m2o^*)$

$= (m2-m1)\cdot[(1-\exp(-t))/(\rho-\exp(-t))] - (n2-n1)\cdot[(1-\rho)/(\rho-\exp(-t))]$

$= (n2-n1)\cdot r\cdot (1-\exp(-t))/(\rho-\exp(-t)) - (n2-n1)\cdot(1-\rho)/(\rho-\exp(-t))$

$= (n2-n1)\cdot\{r\cdot(1-\exp(-t))/(\rho-\exp(-t) - (1-\rho)/(\rho-\exp(-t)\}$

> 0

$\Longleftrightarrow r > (1-\rho)/(1-\exp(-t)).$

The proof of this corollary is completed.

Note: For n2=n1=n, let m1/n=r1, m2/n=r2.

1. m2o ≥ m1o if m2 > m1. This corollary clearly gives us the proof of Lemma 1.2.2. It also tells us that, in the same burn-in facility, more defectives should be screened out from the burn-in lot with more defectives in it.
2. m1 - m1o ≥ m2 - m2o if m2 > m1. In the same burn-in facility, the lot with fewer defectives will be allowed to have more defectives stay in it when burn-in is stopped.
3. $m1o^*/n = \{r1\cdot(1-\exp(-t))-(1-\rho)\}/(\rho-\exp(-t)) < \{r2\cdot(1-\exp(-t))-(1-\rho)\}/(\rho-\exp(-t)) = m2o^*/n$ if m2 > m1. For two burn-in lots with the same lot size, a larger proportion of items (defectives or perfect) has to be eliminated through the course of burn-in from the lot with more defectives.
4. $m1o^*/m1 = (1-\exp(-t))/(\rho-\exp(-t))-(1-\rho)/\{r1\cdot(\rho-\exp(-t))\} < (1-\exp(-t))/(\rho-\exp(-t)) -(1-\rho)\}/\{r2\cdot(\rho-\exp(-t))\} = m2o^*/m2$ if m2 > m1. In the same burn-in facility, a larger portion of defectives must be eliminated through burn-in from the lot with

more defective items in it. The results of 3 and 4 are similar, but they concern different ratios.

Note: For n2 > n1, let m1/n=r1, m2/n=r2.

1. $m1_0 \le m2_0$ if $m1 < m2$. For the lots from the same production line, more defectives must be eliminated through burn-in from the lot which has more defectives at the very beginning of burn-in.
2. $m1 - m1_0 \le m2 - m2_0$ if $m1 < m2$. This means that, for the lots from the same production line, more defectives can stay in the burn-in lot, which has more defectives in it at the beginning of burn-in, when this screen procedure is stopped.
3. $m1_0^*/m1 = (1-\exp(-t))/(\rho-\exp(-t))-(1-\rho)\}/\{r\cdot(\rho-\exp(-t))\} = m2_0^*/m2$ and $m1_0^*/n1 = r\cdot(1-\exp(-t))/(\rho-\exp(-t))\}-(1-\rho)/(\rho-\exp(-t)) = m2_0^*/n2$ if $m2 > m1$. For the burn-in lots from the same production line, almost the same proportion of defectives must be eliminated through burn-in regardless of the size of burn-in lot.

For two positive integers $0 < m1 < m2$, define

$h_{m1}(\Delta) = P(U_{m1_0} < \Delta) = \Sigma_{i=m1_0,m1}\{m1!/[(m1-i)!i!)]\}\Delta^i (1-\Delta)^{m1-i}$ and

$h_{m2}(\Delta) = P(U_{m2_0} < \Delta) = \Sigma_{i=m2_0,m2}\{m2!/[(m2-i)!i!)]\}\Delta^i (1-\Delta)^{m2-i}$

$h_{m1}(\Delta(m1,\alpha,\rho)) = \alpha$ and $h_{m2}(\Delta(m1,\alpha,\rho)) = \alpha$ as in §1.5.

We have the following theorem which clearly tells us the relation between Δ and m if all the other parameters are fixed.

Theorem 1.6.3:

For fixed n, α and ρ,

$$\Delta(m1,\alpha,\rho) < \Delta(m2,\alpha,\rho) \text{ if } m1 < m2. \qquad (1.6.1)$$

Proof:

Define $g(\Delta) = h_{m2}(\Delta)/ h_{m1}(\Delta)$. (1.6.2)

We have

$$g(\Delta) = c \cdot \Delta^{m2o-m1o} \cdot (1-\Delta)^{(m2-m2o)-(m1-m1o)}, \quad (1.6.3)$$

where c is the leading coefficient which is positive.

Since $g(0)=0$, $h_{m2}(0) = h_{m1}(0) = 0$ and $h_{m2}(1) = h_{m1}(1) = 1$, if we can prove that $g(\Delta)$ is a strictly increasing function in Δ for Δ in $[0,1]$, then we prove $h_{m2}(\Delta) < h_{m1}(\Delta)$ for Δ in (0.1). Thus, this theorem is proved, since $h(\Delta)$ is a strictly increasing function in Δ.

$$g'(\Delta)/c = (m2o-m1o) \cdot \Delta^{m2o-m1o-1} \cdot (1-\Delta)^{(m2-m2o)-(m1-m1o)} - \{(m2-m2o)-(m1-m1o)\} \Delta^{m2o-m1o} \cdot (1-\Delta)^{(m2-m2o)-(m1-m1o)-1} \quad (1.6.4)$$

$$= \Delta^{m2o-m1o-1} \cdot (1-\Delta)^{(m2-m2o)-(m1-m1o)-1} \cdot \{(m2o-m1o) \cdot (1-\Delta) - [(m2-m2o)-(m1-m1o)] \cdot \Delta\}$$

$$\geq \Delta^{m2o-m1o-1} \cdot (1-\Delta)^{(m2-m2o)-(m1-m1o)-1} \cdot \{(m2o-m1o) \cdot (1-\Delta)\}, \quad (1.6.5)$$

by using the first part of Corollary (1.6.1), since

$(m2-m2o)-(m1-m1o) \leq 0$ and $m2o - m1o > 0$. So,

$$g'(\Delta) > 0 \iff 0 < \Delta < 1. \quad (1.6.6)$$

Hence $g'(\Delta)$ is a strictly increasing function in Δ for Δ in $(0,1)$.

The proof of this theorem is completed.

Note: In the same burn-in facility, this theorem can be used to compare the durations of burn-in for the production lots from different production lines. It tells us that a longer duration of burn-in is required for the production lot with more defectives in it.

§1.7 The Relation between Δ (∂) and m When m/n is a Constant.

For any production lot of an electronic component from a production line, the ratio of the number of defectives over the size of this lot is usually assumed a constant, r, which is unknown but a suitable value of it is used. In addition, the size of the burn-in facility, n*, may be given. Therefore, it is very important for us to discuss the relation between Δ and (m,n) pair when m/n is a constant, and to see if we can find a sequence of appropriate (m,n) pairs with these n's less than n*, n is the least integer greater than or equal m/r, which will guarantee a relative smaller Δ if this rule is used. This will be explained in the following part of this section. The reason for us to study this is that, in designing a burn-in scheme, we'd like not only to achieve our reliability but also to reduce the cost (or duration) of burn-in as much as possible.

If we can prove that Δ is a monotonic function in m, then we can obtain an appropriate m and its corresponding n to fit the time and the burn-in facility constraint without any difficulty. But, here, Δ is not a monotonic function of m as in §1.6. For fixed α and ρ, if we plot $\Delta(m, \alpha, \rho)$ against m, then we will have a jagged/uneven curve as in the following figure. In this figure, the circled values stand for the "m"s with the same m_0.

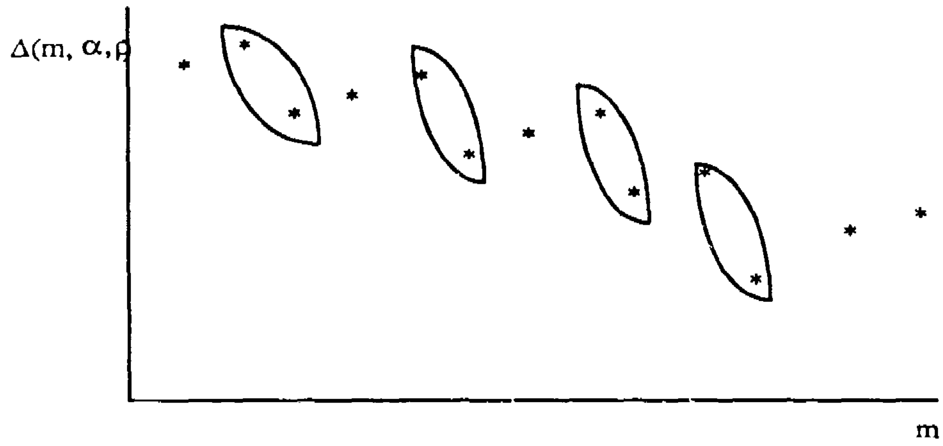

Figure 3

Let's investigate $\Delta(m) = \Delta(m,\alpha,\rho)$ analytically. From § 1.5, we know that

$\Delta(m)$ is the solution of $\sum_{i=m_0,m}\{m!/[(m-i)!i!)]\}\Delta^i (1-\Delta)^{m-i} = \alpha$. (1.7.1)

In addition, for any positive integer m and $0 < r < 1$, we have

$m_0^* = (m(1-\exp(-t))-n(1-\rho))/(\rho-\exp(-t))$ and (1.7.2)

$m_0 = [m_0^*]+1$ if $m_0^* > [m_0^*]$ or $m_0 = m_0^*$ if $m_0^* = [m_0^*]$. (1.7.3)

Define n as a function of m: $n(m) = [m/r] + 1$ if $m/r > [m/r]$ or $n(m) = [m/r]$ if $m/r = [m/r]$.

So, for any positive integer k, we have

$(m+k)_0^* = ((m+k)\cdot(1-\exp(-t))-n(m+k)\cdot(1-\rho))/(\rho-\exp(-t))$

$\sim (m\cdot(1-\exp(-t))-n(m)\cdot(1-\rho))/(\rho-\exp(-t)) + k\cdot((1-\exp(-t))-(1/r)\cdot(1-\rho))/(\rho-\exp(-t))$

$= m_0^* + k\cdot((1-\exp(-t))-(1/r)\cdot(1-\rho))/(\rho-\exp(-t))$

$= m_0^* + k\cdot m_0^*/m = m_0^*\cdot(1+k/m)$.

In addition, there is an integer $k1 \geq 0$ such that, for $0 \leq k \leq k1$,

$(m+k)_0 = m_0$

$(m+k1)_0 = m_0$ and

$(m+k1+1)_0 = m_0 + 1$.

If $k1 \geq 1$ and $(m+k1)_0 = m_0$, let's compare the solution of

$$\alpha = \Sigma_{i=m0,m}\{m!/[(m-i)!i!)]\}\Delta^i (1-\Delta)^{m-i} \quad (1.7.4)$$

and the solution of

$$\alpha = \Sigma_{i=(m+k1)0,(m+k1)}\{(m+k1)!/[((m+k1)-i)!i!)]\}\Delta^i (1-\Delta)^{m+k1-i}$$
$$= \Sigma_{i=m0,(m+k1)}\{(m+k1)!/[((m+k1)-i)!i!)]\}\Delta^i (1-\Delta)^{m+k1-i}. \quad (1.7.5)$$

If we let Δ be the probability of failure, then (1.7.4) is the probability of at least m_0 failures in m trials and (1.7.5) is the probability of at least m_0 failures in m+k1 trials. If both of (1.7.4) and (1.7.5) are equal to α, it is clear that the solution, $\Delta(m)$, for (1.7.4) is greater than that of (1.7.5), $\Delta(m+k1)$.

Similarly, if $k1 \geq 1$ and $(m+k1)_0 = m_0+1$, let's compare the solution of

$$\alpha = \Sigma_{i=m0,m}\{m!/[(m-i)!i!)]\}\Delta^i (1-\Delta)^{m-i} \quad (1.7.6)$$

and the solution of

$$\alpha = \Sigma_{i=(m+k1)0,(m+k1)}\{(m+k1)!/[((m+k1)-i)!i!)]\}\Delta^i (1-\Delta)^{m+k1-i}$$
$$= \Sigma_{i=m0+1,(m+k1)}\{(m+k1)!/[((m+k1)-i)!i!)]\}\Delta^i (1-\Delta)^{m+k1-i}. \quad (1.7.7)$$

If $k1 = 1$, using the same argument as the above, we have $\Delta(m) < \Delta(m+1)$. If $k1 \geq 2$, the relation between $\Delta(m)$ and $\Delta(m+1)$ is not quite clear. Summarizing these results, we have the following lemma.

Lemma 1.7.1:

If $(m+1)_0 = m_0$, then $\Delta(m) > \Delta(m+1)$.

If $(m+1)_0 = m_0+1$, then $\Delta(m) < \Delta(m+1)$.

Using Lemma 1.7.1, we can simplify our search for the minimum $\Delta(m)$'s for all $m \leq n \cdot r$. So, this lemma can be used to help us in setting the most economic lot size of the burn-in facility if an assumed r is used and the maximum possible lot size is given.

Note: We might be interested in the average duration of burn-in per item rather the duration of burn-in.

§1.8 Additional Relations between m and the Stopping Time ∂.

This stopping rule is a fixed time stopping rule which is different from all the other three (sequential) rules which will be discussed in the following chapters. The possible available information about m can be used to improve the accuracy of ∂ (or Δ) as mentioned in the above. ∂ (or Δ) can be derived easily, according to each case presented below about the available information about m, by following the results mentioned in §1.3 and §1.5.

For an assumed value of m, say m^e, let $r^e=m^e/n$. Based on the available information about m, we have

$m^{e*}_0 = [m^e(1-\exp(-t))-n(1-\rho)]/(\rho-\exp(-t))$, and (1.8.1)

$s_{m^e0}=m^e_0/m^e=m^{e*}_0/m^{e*}=\{[1-\exp(-t)] - (1/r^{e*})\cdot(1-\rho)\}/(\rho\cdot\exp(-t))$ if $m^e/n=r^e$. (1.8.2)

Following (1.3.11), we have

$\partial^e = -\ln(1- s_{m^e0} - z_\alpha \sqrt{[s_{m^e0}(1-s_{m^e0})]} / \sqrt{m^e})$. (1.8.3)

As discussed in §1.4, when m^e is sufficiently large, ∂^e mainly depends on s_{m^e0}. Hence, if the binary search algorithm, §1.6, is used in finding Δ (or ∂), then Δ (or ∂) mainly depends on $m-m_0$, rather than on m.

As in Lemma 1.2.3 and Theorem 1.6.3, if m^e is larger than true m, then the duration of burn-in is longer than what is truly needed, i.e., $\partial^e > \partial$. This is the case when m is over-estimated. If m^e is smaller than true m, then we might not be able to achieve our reliability goal $P(R(t; D=\partial^e, m, n) \geq \rho) \geq \alpha$.

The duration of this stopping rule is ∂, a fixed constant. Hence the expected duration of burn-in is ∂. This is not the same as the duration of the other three stopping rules, as they are random, which will be seen in the following chapters.

When an estimate of m, m^e, is used in this stopping rule, the assumed reliability of a randomly chosen item from an after-burn-in production lot is

$$P(T_1 < \partial, T_2 < \partial, ..., T_{m^e 0} < \partial) = P(T_{m^e 0} < \partial) \quad (1.8.4)$$

which is at least α according to equation(1.3.6) and Lemma 1.2.3 if m^e is at least true m.

Note: the probability (1.8.4) depends on true m. In this case, when an upper bound of m is used, a lower bound of this probability, $P(T_1 < \partial, T_2 < \partial, ..., T_{m0} < \partial)$, is obtained, since a larger portion of defectives could be eliminated. More accurate value of this probability can be obtained if more accurate information about m is available.

If $M \sim P(M=m|\theta)$ is the prior distribution of m, to achieve $P(R(t; D, M, n) \geq \rho) \geq \alpha$ by using this screen procedure, we need to find the ∂ with

$$\sum_{n \cdot \{(1-)/(1-\exp(-t))\} \leq m \leq n} P(T_{m0} < \partial) \cdot P(M=m|\theta) + P(M < n \cdot \{(1-\rho)/(1-\exp(-t))\}) = \alpha,$$

or to find the corresponding $\Delta = 1 - \exp(-\partial)$ with

$$\sum_{n \cdot \{(1-)/(1-\exp(-t))\} \leq m \leq n} P(U_{m0} < \Delta) \cdot P(M=m|\theta) + P(M < n \cdot \{(1-\rho)/(1-\exp(-t))\}) = \alpha.$$

The left hand side of the above equation is clear to us which is an increasing function in Δ with bounded first derivative for Δ in [0,1]. Hence, the binary search algorithm defined before can be used to find the appropriate Δ with the desired level of accuracy.

§1.9 The Number of Defectives Left after Burn-in

Denote L_∂ as the number of defectives left in the burn-in lot when this burn-in procedure is stopped with duration ∂. We know that $L_\partial=m-J_\partial$, the number of defectives left after burn-in, is the number of defectives at the beginning of burn-in minus the number of defectives been screened out during burn-in. Marcus and Blumenthal (1974) have a detailed study of $P(L_\partial \leq \varsigma)$, where ς is the allowed maximum number of defective left after burn-in being stopped, about the case that m is unknown. Their rule is conservative.

For the case m is given, the probability that the number of defectives left, after burn-in being stopped, will not exceed ς, for any given ς, is

$P(L_\partial < \varsigma | m)$

$$= P(J_\partial \geq m-\varsigma \,|m) \tag{1.9.1}$$

$$= \Sigma_{j=m-\varsigma,m} P(J_\partial=j|m). \tag{1.9.2}$$

To solve this, we have $P(J_\partial=j|m)$

$$=P(T_j < \partial|m) - P(T_{j+1} \leq \partial|m) \tag{1.9.3}$$

$$= P(T_j < \partial, T_{j+1} > \partial|m) \tag{1.9.4}$$

$$= P(U_j < 1-\exp(-\partial), U_{j+1} > 1-\exp(-\partial) \mid m) \text{ where } U_i=1-\exp(-T_i) \text{ for } i=j,j+1 \tag{1.9.5}$$

$$= \{(m!)/[(j-1)!(m-j-1)!]\}\int_{0,\,1-\exp(-\partial)}\int_{1-\exp(-\partial),1}(u_j)^{j-1}(1-u_{j+1})^{m-j-1}\,du_j du_{j+1}$$

$$= \{(m!)/[j!(m-j)!]\}[1-\exp(-\partial)]^j[\exp(-\partial)]^{m-j}. \tag{1.9.6}$$

It is a well-known that $J_\partial \sim$ binomial$(m,1-\exp(-\partial))$. To stop at $J_\partial=j$ means that j failed defectives were observed before ∂. The probability for the occurrence of any failure before ∂ is $1-\exp(-\partial)$. Hence the distribution of J_∂ should be binomial. From (1.9.1) and (1.9.2), we have that $P(L_\partial < ç|m)$ is a partial sum of the above binomial distribution. This probability can be calculated directly or obtained easily by using a table of binomial distributions or using normal approximation if m is large enough or compute it directly as before.

If larger m is used as its true value, a conservative rule is used, then, by Lemma 1.2.3, the true value of (1.9.2) will be more than what is calculated.

If m has a prior distribution, say $M \sim P(M=m|\theta)$ for $m=0,1,\cdots,n$, then we have

$$
\begin{aligned}
&P(L_\partial < ç) \\
&= \Sigma_{m=0,n} P(M=m|\theta)\cdot P(J_\partial \geq m-ç\ |m) \\
&= \Sigma_{m=0,n} P(M=m|\theta)\cdot \Sigma_{j=m-ç,m} P(J_\partial=j|m) \\
&= \Sigma_{m=0,ç} P(M=m|\theta) + \Sigma_{m=ç+1,n} P(M=m|\theta)\cdot \Sigma_{j=m-ç,m} P(J_\partial=j|m) \qquad (1.9.7)
\end{aligned}
$$

Given the prior distribution of m, (1.9.7) is the probability that the total number of defectives left after burn-in will not exceed the specified upper bound.

We can use (1.9.7), let it be α, and use the binary search algorithm defined before to find the appropriate ∂, $\partial=-\ln(1-\Delta)$. This ∂ ensures that with probability α the number of defectives remaining will not exceed a given bound, when, $M \sim P(M=m|\theta)$, the prior distribution of m is given.

The expected number of defectives left after burn-in can be derived,too. For the case m is given, we have

$$E(J_\partial|m) = m\cdot[1-\exp(-\partial)]. \qquad (1.9.8)$$

If $M \sim P(M=m|\theta)$, we have

$$E(J_\partial) = \Sigma_{m=0,n} P(M=m|\theta) \cdot E(J_\partial|m). \tag{1.9.9}$$

If M ~ binomial(n,r), then

$$E(J_\partial) = \Sigma_{m=0,n} \{n!/[(n-m)!(m!)]\}(r)^{m}(1-r)^{n-m} \cdot m\,[1-\exp(-\partial)]. \tag{1.9.10}$$

This is the expected number of defectives left after burn-in if the number of defectives has the binomial prior.

§1.10 $E(R(t;\ D=\partial,\ m,\ n))$

We already have

$R(t; D=\partial, M, n) = 1 - [(M-J_\partial)/(n-J_\partial)]\cdot[1-\exp(-t)]$.

(1.10.1)

Hence, if we are given m, then

$E\{R(t; D=\partial, M=m, n)\} = E\{1 - [(m-J_\partial)/(n-J_\partial)]\cdot[1-\exp(-t)]\}$

$= \sum_{m0\le j\le m} \{1 - [(m-j)/(n-j)]\cdot[1-\exp(-t)]\}\cdot P(J=j|m) + P(J<m0)$

$= \sum_{m0\le j\le m} \{1-[(m-j)/(n-j)]\cdot[1-\exp(-t)]\}\cdot \{(m!)/[j!(m-j)!]\}[1-\exp(-\partial)]^j[\exp(-\partial)]^{m-j}$

$+ P(J<m0)$. (1.10.2)

This is the expected reliability after a fixed duration of burn-in, ∂, when m is given. It is obtained by summing over all possible reliabilities weighted by their corresponding probabilities. If m is overestimated, by lemma 1.2.3, this reliability will be more than what we thought.

If $M \sim P(M=m|\theta)$ for $m=0,1,\cdots,n$, then

$E(R(t; D=\partial, M, n))$

$= \sum_{m=0,n} P(M=m|\theta)\cdot E\{R(t; D=\partial, M=m, n)\}$

$=\sum_{m=0,n}P(M=m|\theta)\cdot\{\sum_{m0\le j\le m}\{1-[(m-j)/(n-j)]\cdot[1-\exp(-t)]\}\cdot\{(m!)/[j!(m-j)!]\}\cdot$

$[1-\exp(-\partial)]^j\cdot[\exp(-\partial)]^{m-j} + P(J<m0)\}$.

(1.10.3)

This the expected reliability after burn-in if this stopping rule is used and $M \sim P(M=m|\theta)$.

If we let $E\{R(t; D=\partial, M=m, n)\}=\alpha$ in (1.10.2) and solve it for ∂, this stopping rule will guarantee $E(R(t; D=\partial, M=m, n))=$(or $\geq$)α, when we know (or overestimate) the value of m. If we let $E(R(t; D=\partial, M, n))=\alpha$ in (1.10.3) and solve it for ∂, this stopping rule will guarantee $E(R(t; D=\partial, M, n))=\alpha$, when the prior distribution of m, $M \sim P(M=m|\theta)$, is given.

CHAPTER II

PROCEDURE I

§2.1 Introduction

The reliability goal $P(R(t; D, m, n) \geq \rho) \geq \alpha$ can be achieved by screening out some portion of the defectives from the production lot which is under burn-in as described in Procedure 0. By considering the waiting time between failures, W_i, where

$W_i = T_i - T_{i-1}$ and $T_0 = 0$, we get the procedure I. Burn-in is never terminated until the first W_i exceeds some given bound. The screening procedure, Procedure I, is based on "A Sequential Screening Procedure" by Marcus and Blumenthal in Technometrics (1974). The goal of their paper is to develop a sequential procedure in order to screen out the defective items through "burn-in" such that the number of the remaining defectives under burn-in at the time of stopping is bounded by a given constant with a pre-specified probability level. This idea will be modified to develop Procedure I. In addition, the key idea here is that $1/E(W_i)$ is monotonically increasing with respect to i or $P(W_i < t^*)$ decreases as i is increased for any given positive t^*.

The following is a brief summary of the contents of this chapter. Section two is a description of the stopping rule developed in this chapter and the definition of t^*. A general search algorithm for t^* is given in section three and another search algorithm for the t^* defined in Marcus and Blumenthal (1974) is given in section four. Sections five and six consider the relation between t^* and m under two different conditions. Sections seven, eight and nine discuss the computation schemes to obtain t^* based on the different cases about the available information regarding m. The probability distribution of this stopping rule is given in section ten when the value of m is known. The expected duration of burn-in is one of the most important criteria to judge the performance of all the stopping rules developed in this thesis. The computation of the expected duration of this stopping rule is given in section eleven. The behavior of the expected duration of this stopping rule is discussed in section thirteen to fifteen under several different conditions. The last section is a brief comparison between this stopping rule and the similar rule developed in Marcus and Blumenthal (1974).

§2.2 The Stopping Rule

Let $0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_m$ be the ordered sequence of times to failure of the defective items under burn-in. Let $W_i = T_i - T_{i-1}$ for $i=1,2, \cdots ,m$ be the waiting times between failures of the defective items. Let J (or J_D) be the number of failed defectives observed during burn-in when the duration of burn-in is D.

In order to ensure

$$R(t; D, m, n) \geq \rho, \tag{2.2.1}$$

we must have $J \geq m_0$, where

m_0 = the least integer greater than or equal to

$$\{m\cdot(1-\exp(-t))-n\cdot(1-\rho)\}/(\rho-\exp(-t)), \tag{2.2.2}$$

with a pre-specified probability, α , as discussed in lemma 1.2.1. We also hope that this screening procedure can be stopped as soon as J reaches m_0 (when an assumed m is given). The goal (2.2.1) can be achieved by finding a t* such that

$$W_1 < t^*, W_2 < t^*, \ldots , W_{m_0} < t^* \text{ with probability at least } \alpha. \tag{2.2.3}$$

This is the same as to find a t* such that, under the stopping rule described below,

$P(R(t; D, m, n) \geq \rho) \geq \alpha$.

Stopping Rule I:

Stop burn-in at the time when the first j is reached with $W_j \geq t^*$. (S.2.1)

Under this stopping rule, formulating (2.2.3), we have

$$P(W_1 < t^*, W_2 < t^*, \ldots , W_{m_0} < t^*) = P(J > m_0) \quad \geq \quad \alpha . \tag{2.2.4}$$

Actually, since m is unknown, an appropriate estimate of m is used in evaluating (2.2.4). The relation between m and t^* will be studied in more detail from §2.5 to §2.9.

Note: In Marcus and Blumenthal (1974), they found the t^* to ensure $P(m\text{-}J \leq ç) \geq \alpha$ for some given fixed ç when the information about m is assumed to be unavailable. Here, we try to find the smallest t^* with

$$P(m-J \leq m-m_0) \geq \alpha \qquad (2.2.5)$$

when some information about m is available. Inside the parenthesis of (2.2.5), the right- hand side, $m-m_0$, is their ç. In evaluating the above probability, we should be very careful since the exact value of m is not available. We can derive a lower bound of this probability based on the available information about m and guarantee that the true probability is greater than this lower bound.

§2.3 A Binary Search Algorithm for t*

Since T_i for i=1,2, ...,m is an order statistic from a population of size m (assumed) with standard exponential distribution, it is well-known that W_i for i=1,2, ..., m are independent and exponentially distributed with c.d.f.

$$1-\exp(-(m+1-i)\cdot w) \tag{2.3.1}$$

(Pyke (1965) or Sukhatme (1937)). To save the amount of time on burn-in, let's consider the equality case of (2.2.4) only:

$$P(W_1 < t^*, W_2 < t^*, \ldots, W_{m0} < t^*) = \alpha . \tag{2.3.2}$$

In addition,

$$P(W_1 < t^*, W_2 < t^*, \ldots, W_{m0} < t^*)$$

$$= \prod_{1\le i\le m0} (1-\exp(-(m-i+1)\cdot t^*))$$

$$= \prod_{m-m0+1\le i\le m}(1-\exp(-i\cdot t^*)) \tag{2.3.3}$$

$$\ge 1-\sum_{m-m0+1\le i\le m}(\exp(-i\cdot t^*))$$

$$\ge 1-\{\exp(-(m-m0+1)\cdot t^*)-\exp(-(m+1)\cdot t^*)\}/(1-\exp(-t^*)) \tag{2.3.4}$$

$$\ge 1-\exp(-(m-m0+1)\cdot t^*)/(1-\exp(-t^*)) \tag{2.3.5}$$

All the above formulae are given in Marcus and Blumenthal (1974) as mentioned before. Expression (2.3.5) is extensively studied by them, as well. Most of their results can be applied to this screening procedure for the case when no information about m is available. If the left hand side of (2.3.2) is replaced by (2.3.5), the solution, t*, of (2.3.2) will guarantee that the number of defectives left after burn-in will not exceed a given upper bound, ç, with probability at least α under this sequential screening procedure. That is to replace m-m0 by ç which is independent of the unknown m. Note that m-mo is the number of defectives that may remain in the burn-in lot after burn-in. This is the advantage of their rule. However, any information about m, which could be very valuable, is thrown away in this case.

In the evaluation of (2.3.5), Marcus and Blumenthal, had a very complicated computation scheme. In order to simplify the computation in calculating t*, we need the following lemma.

Lemma 2.3.1:

Define, for any two positive integers k and n with $1 \leq k < n$,

$$g1(x)=\prod_{k\leq i\leq n}(1-\exp(-i\bullet x)), \qquad (2.3.6)$$

$$g2(x)=1-\{\exp(-k\bullet x)-\exp(-(n+1)\bullet x)\}/\{1-\exp(-x)\} \text{ and} \qquad (2.3.7)$$

$$g3(x)=1-\exp(-k\bullet x)/(1-\exp(-x)). \qquad (2.3.8)$$

Here, g1(x) and g2(x) are increasing functions in x with bounded and positive first derivatives for x in $[0,\infty)$. In addition, g3(x) is an increasing function in x with bounded and positive first derivative for x in $[\varepsilon,\infty)$ where ε, nonegative, is the solution of g3(x)=0.

Note: Equations (2.3.3), (2.3.4) and (2.3.5) are the special cases of the equations (2.3.6), (2.3.7) and (2.3.8), respectively. The t* derived by solving $g1(t^*)=\alpha$ or $g2(t^*)=\alpha$ can be any number between 0 and ∞, but the t* derived by solving $g3(t^*)=\alpha$ is bounded away zero from below. Function g3(x) is nonnegative if $x < \varepsilon$. The t* solved by using g3 is the most conservative one.

Proof:

g1(x) is the product of some strictly increasing functions in x with bounded and positive first derivative for x in $(0, \infty)$. So, g1(x) is an increasing function in x with bounded and positive first derivative for x in $(0, \infty)$.

$g2(x)=1-\{\exp(-k\cdot x)-\exp(-(n+1)\cdot x)\}/\{1-\exp(-x)\}$

$= 1 - \sum_{i=k,n}\exp(-i\cdot x)$.

So, it is clear to us that g2(x) is a strictly increasing function in x with bounded and positive first derivative for x in $(0,\infty)$.

Similarly, for g3(x), we have $\lim_{x\to 0} g3(x) = -\infty$ and $g3(\infty) = 1$. In addition,

$g3'(x) = \{k\cdot\exp(-kx)\cdot(1-\exp(-x)) + \exp(-kx)\cdot\exp(-x)\}/\{1 - \exp(-x)\}^2 > 0$

for all $x \geq 0$. Hence, g3(x) is strictly increasing in $[0.\infty)$ and there is a unique ε in (0,1) such that $g3(\varepsilon) = 0$. Moreover, for all x in $[\varepsilon,\infty)$, we have $g3'(x) \leq (k+1)/\{1 - \exp(-\varepsilon)\}^2 < \infty$.

The proof of this lemma is completed.

This lemma implies that (2.3.3), (2.3.4) and (2.3.5) are monotonically increasing functions of t*. We have the following corollary.

Corollary 2.3.1

1) Let k=m-m0+1 and n=m in g1(t*), we have (2.2.8) is a monotonically increasing function in t* for t* in $(0, \infty)$ with a bounded and positive derivative.

2) Let k=m-m0+1 and n=m in g2(t*), we have (2.2.9) is a monotonically increasing function in t* for t* in $(0,\infty)$ with a bounded and positive derivative.

3) Let k=m-m0+1, we have (2.2.10) is a monotonically increasing function in t* for t* in (ε,∞) with a bounded and positive derivative, where ε is defined in Lemma 2.3.1.

Using the above results, the expressions (2.3.3), (2.3.4) and (2.3.5) are monotonically increasing functions in t* with ranges in $[0,\infty]$, we can easily use a binary search to find t^*, as we did in finding Δ in procedure 0, if we let these expressions equal α.

A Binary Search Algorithm for Finding t^*:

Let $y = \exp(-t^*)$ and let expression (2.3.3) (or (2.3.4), or (2.3.5)) = $h(y)$.

1) Let $y_1=1/2$.

2) If $h(y_i) > \alpha$, then $y_{i+1} = y_i - (1/2)^{i+1}$.

   If $h(y_i) < \alpha$, then $y_{i+1} = y_i + (1/2)^{i+1}$.

   If $h(y_i) = \alpha$, then $y_{i+1} = y_i$.

3) Stop, if $|y_{i+1} - y_i| \le e$, where e is a given error bound.

4) Let $y^*=y_{i+1}$ and $t^*=-\ln(y^*)$.

After 30 iterations, we will have $|y_{30} - y^{**}| < 10^{-9}$ where $y^{**} = \exp(-t^{**})$ and t^{**} is the solution of the corresponding equation. In addition, it is clear that the solution for (2.3.3) is less than that for (2.3.4), and both of them are less than the solution for (2.3.5). From Theorem 1.2.2, we can prove that the sequences of values found through this binary search converge to the solutions of the corresponding equations if we can prove that

$h1(y)=\prod_{k\le i\le n}(1-y^i)$ and (2.3.9)

$h2(y)=1-\{y^k-y^{n+1}\}/\{1-y\}$ (2.3.10)

are differentiable with bounded negative derivative for y in (0,1), and

$h3(y)=1 - y^k/(1-y)$. (2.3.11)

is differentiable with bounded negative derivatives for y in $(0,\varepsilon^*)$ where $\varepsilon^*=\exp(-\varepsilon)$. This is another corollary of Lemma 2.3.1.

Corollary 2.3.2

Functions h1(y) and h2(y) are differentiable with bounded negative derivatives in (0,1). In addition, h3(y), for any ε in (0,1), is differentiable with a bounded negative derivative in $(0,\varepsilon)$. So, by Theorem 1.2.2, we have proved that the above binary search algorithm in finding t* is convergent to the unique true t*.

§2.4 A Fixed Point Iterative Algorithm to Find the Most Loose Bound for t*

$g3(t^*) = \alpha.$

$\Leftrightarrow 1\text{-}\exp(-k \cdot t^*)/(1-\exp(-t^*)) = \alpha.$

$\Leftrightarrow (1 - \alpha) \cdot (1 - \exp(-t^*)) = \exp(-k \cdot t^*)$

$\Leftrightarrow t^* = -\ln(1 - \alpha)/k - \ln(1 - \exp(-t^*))/k.$ (2.4.1)

So, we can also use a fixed point iterative algorithm to find t* , by using (2.4.1) as described below:

A Fixed Point Iterative Algorithm to Find t* by Using (2.4.1):

1) Let $t_0 = -\{\ln(1 - \alpha)\}/k$.
2) Let $t_i = t_0 - \{\ln(1 - \exp(-t_{i-1}))\}/k$ for $i=1,2,\ldots$
3) Stop when $| t_i - t_{i-1} | \le e$, where e is a prespecified error bound.
4) Let $t^*=t_i$.

The two fixed point algorithms in Section 2.7 and Section 2.8 will be used to compare the duration of burn-in between the case that no information about m is available (m=n-1) and the case that a smaller upper bound of m is available (m<n-1).

Let $g(x)=-\ln(1-\alpha)-\ln[1-\exp(-x)]$ for x in $(0,\infty)$ and α in (0,1). For any $\mu \ge 1$, let $h(x)=g(x)/\mu$. The following lemma proves that the above iterative algorithm is convergent.

Lemma 2.4.1: (Convergence of a Fixed Point Iterative Algorithm)

For any $\mu \geq 1$ and any α in $(1-\{\mu/(\mu+1)\}^{\mu},1)$, the following fixed point iterative algorithm is convergent:

1) Let $x_0 = h(0) = -\ln(1-\alpha)/\mu$.
2) Let $x_i = h(x_{i-1}) = \{-\ln(1-\alpha)-\ln[1-\exp(-x_{i-1})]\}/\mu$ for i=1,2,...
3) Stop when $| x_i - x_{i-1} | \leq e$, where e is a prespecified error bound.
4) Let $x^* = x_i$.

Proof:

It is trivial that $h(x_i) \geq h(x_0)$ for i=0,1,... and h(x) is a continuous and differentiable function which maps from $[x_0,\infty)$ into $[x_0,\infty)$. In addition, $h'(x) = -\exp(-x) /[1-\exp(-x)]\bullet\mu\}$. Using Theorem 3.1 (page 90) of Contel & de Boor(1980), we only need to prove that $|h'(x)| < 1$ for all x in $[x_0,\infty)$ if α in $(1-\{\mu/(\mu+1)\}^{\mu}, 1)$. We have

$|h'(x)| < 1$

$\Longleftrightarrow [\exp(-x) / [1-\exp(-x)]]/\mu < 1$

$\Longleftrightarrow \exp(-x) < \mu\bullet[1-\exp(-x)]$

$\Longleftrightarrow [1+\mu]\bullet\exp(-x) < \mu$

$$\Longleftrightarrow x > -\ln(\mu/[1+\mu]). \qquad (2.4.2)$$

So, we need $x_0 > -\ln(\mu/[1+\mu])$ to ensure $|h'(x)| < 1$for all x in $[x_0,\infty)$.

Moreover, $x_0 = -\ln(1-\alpha)/\mu$.

Hence, $x_0 > -\ln(\mu/[1+\mu])$

$\Longleftrightarrow -\ln(1-\alpha)/\mu > -\ln(\mu/[1+\mu])$

$\Longleftrightarrow 1-\alpha < (\mu/[1+\mu])^{\mu}$

$$\Longleftrightarrow \alpha > 1- (\mu/[1+\mu])^{\mu}. \qquad (2.4.3)$$

In addition, the upper bound of α is 1.

In addition, the upper bound of α is 1.

The proof of this lemma is completed.

Note: For $\mu \geq 1$, we have $\alpha \geq 0.5$. We are typically interested in the case that α is close to 1 in considering P(R(t; D, m, n) $\geq \rho$) $\geq \alpha$. Hence, $\alpha \geq 0.5$ is good enough.

Corollary 2.4.1:

For any m, $0 \leq m \leq n-1$, $\mu = (n-1) - m_0 + 1$, and the allowable range for α is

$$1 > \alpha > 1 - \{ [(n-1) - m_0 + 1]/\{1 + [(n-1) - m_0 + 1]\} \}^{(n-1) - m_0 + 1} . \qquad (2.4.4)$$

In addition, for $m_0 \leq n-1$,

$$1/2 \geq 1 - \{ [(n-1) - m_0 + 1]/\{1 + [(n-1) - m_0 + 1]\} \}^{(n-1) - m_0 + 1} \qquad (2.4.5)$$

The equality of (2.4.5) is true only when $(n-1) = m_0$.

The t*'s solved in §2.3 and §2.4 where we assumed that the true value of m is known or an appropriate estimate of it is used. For solving P($W_1 < t^*$, $W_2 < t^*$, •••, $W_{m0} < t^*$) = Equation (2.3.3) = α or Equation (2.3.4)=α or Equation (2.3.5)=α, different m's (estimated sizes of defectives) will produce different t*'s. In the following sections, we'll study the relationship between t* and m when n is fixed.

§ 2.5 The Relation between t^* and m When n, α Is Fixed.

For a fixed n, define $t^*(\alpha,m)$ as the solution of (2.3.3) or (2.3.4) or (2.3.5) where m is the assumed number of defective items. Recall the result of Lemma 1.2.3 in Procedure 0, in order to ensure $P(R(t;\ D,\ m,\ n)\geq\rho)\geq\alpha$, we use the (least) upper bound of m, say ${}^u m$, as the true m, so that a lager portion of defectives must be eliminated before the screening procedure is stopped and a higher probability, $P(R(t;\ D,\ m,\ n)\geq\rho)$ is obtained. So, we can use the least available upper bound of m, ${}^u m$, as its true value and solve (2.3.2), by using (2.3.3) or (2.3.4) or (2.3.5), to derive the corresponding $t^*(\alpha,{}^u m)$'s. Now, we face a very crucial problem. Do these $t^*(\alpha,{}^u m)$'s truly guarantee $P(R(t\ (\alpha,{}^u m);\ D,\ m,\ n)\geq\rho)\geq\alpha$, where m is its true value, if this screening procedure is used?

Before solving this problem, let m1 and m2 be two positive integers, m1 < m2, with values less than two integers n1 and n2, respectively, n1 < n2, and let

$mi_0^* = [mi\cdot(1\text{-}\exp(\text{-}t))\text{-}n\cdot(1\text{-}\rho)]\ /\ [\rho\text{-}\exp(\text{-}t)]$ and (2.5.1)

mi_0 = the least integer greater than or equal

$$[mi\cdot(1\text{-}\exp(\text{-}t))\text{-}n\cdot(1\text{-}\rho)]\ /\ [\rho\text{-}\exp(\text{-}t)] \qquad (2.5.2)$$

where i=1 or 2 (as defined in §1.6). The following theorem helps us in solving this problem.

Theorem 2.5.1:

For n1 = n2 = n, given a fixed $t^* > 0$ and m1 < m2,

$P(\ W_1 < t^*,\ W_2 < t^*,\ \ldots\ ,\ W_{m1_0} < t^* | m{=}m1\ )$

$> P(\ W_1 < t^*,\ W_2 < t^*,\ \ldots\ ,\ W_{m2_0} < t^* | m{=}m2\ ),$ (2.5.3) where W_i is the waiting time between the i-1st and the ith failure as defined before and

$P(W_1 < t^*, W_2 < t^*, \ldots, W_{mio} < t^* | m=mi)$ denote $P(W_1 < t^*, W_2 < t^*, \ldots, W_{mio} < t^*)$ when the true value of m is mi for i=1,2.

Note: This theorem tells us if the same t* is used for two different burn-in lots with their sizes of defectives being m1 and m2, respectively, in the same burn-in facility (same n) then the lot with fewer defectives in it will have higher probability to achieve the reliability goal, $P(R(t; D, m, n) \geq \rho) \geq \alpha$, when this screen procedure is used with the same t*.

Proof:

$$P(W_1 < t^*, W_2 < t^*, \ldots, W_{m1o} < t^* | m=m1)$$

$$= \prod_{m-m1o+1 \leq i \leq m1} (1 - \exp(-i \cdot t^*)).$$

$$P(W_1 < t^*, W_2 < t^*, \ldots, W_{m2o} < t^* | m=m2)$$

$$= \prod_{m-m2o+1 \leq i \leq m2} (1 - \exp(-i \cdot t^*)) .$$

Using Corollary 1.6.1 in Chapter I, we have

$$(m1-m1o+1)-(m2-m2o+1)$$

$$= (m1-m1o)-(m2-m2o) \geq 0, \tag{2.5.4}$$

since $m1 \leq m2$. This inequality, (2.5.4), implies that

$$\prod_{m2-m2o+1 \leq i < m2} (1 - \exp(-it^*)) \leq \prod_{m1-m1o+1 \leq i < m1} (1 - \exp(-it^*)) . \tag{2.5.5}$$

This implies that

$$P(W_1 < t^*, W_2 < t^*, \ldots, W_{m2o} < t^* | m=m2)$$

$$\leq P(W_1 < t^*, W_2 < t^*, \ldots, W_{m1o} < t^* | m=m1). \tag{2.5.6}$$

Hence we complete the proof of this theorem.

Based on this theorem, if two estimates of m, $m1 < m2$, and (2.3.3) are used in deriving $t^*(m,\alpha)$ for the same burn-in lot, then the t^* corresponding to the smaller estimate, m1, will have smaller value than the t^* corresponding to the larger estimate, m2. So, we have the following corollary. This is a trivial result of (2.5 5).

Corollary 2.5.2:

For $n1=n2=n$, let $t^*(m,\alpha)$ be the solution of

$$\Pi_{m-m0+1\leq i\leq m}(1-\exp(-it^*)) = \alpha. \tag{2.5.7}$$

If $0 < m1 < m2$, then $t^*(m1,\alpha) < t^*(m2,\alpha)$.

This Corollary tells us a conservative rule is used if an upper bound of m is used as its true value in this screening procedure. Based on (2.3.3), for fixed n , ρ and α, $t^*(m,\alpha)$ is a monotonically increasing function in m.

Can we have the results similar to corollary 2.3.2, for the $t^*(m,\alpha)$'s based on (2.3.4) and (2.3.5)? The following two theorems give us the answer.

Theorem 2.5.2

For $n1=n2=n$, let $t^*(m,\alpha)$ be the solution of

$$1 - \{ \exp(-(m-m0+1)\cdot t^*) - \exp(-(m+1)\cdot t^*) \} / (1- \exp(-t^*)) = \alpha. \tag{2.5.8}$$

If $0 < m1 < m2$, then $t^*(m1,\alpha) \leq t^*(m2,\alpha)$.

Proof:

This is the same as to prove, for any $t^* > 0$ and $m1 < m2$,

$$1 - \{ \exp(-(m1-m1_0+1)\cdot t^*) - \exp(-(m1+1)\cdot t^*) \} / (1- \exp(-t^*)).$$

$$> 1 - \{ \exp(-(m2-m2_0+1)\cdot t^*) - \exp(-(m2+1)\cdot t^*) \} / (1- \exp(-t^*)). \tag{2.5.9}$$

The above inequality, (2.5.9), is true (since both sides of (2.5.9) are increasing functions in t^* by Lemma 2.3.1)

$\Leftrightarrow \{\exp(-(m1-m1_0+1)\cdot t^*)-\exp(-(m1+1)\cdot t^*)\}<\{\exp(-(m2-m2_0+1)\cdot t^*)-\exp(-(m2+1)\cdot t^*)\}$

$\Leftrightarrow 1-\exp(-m1_0\cdot t^*)<\exp(-\{(m2-m2_0)-(m1-m1_0)\}\cdot t^*)-\exp(-\{(m2-(m1-m1_0)\}\cdot t^*)$

$$\Leftrightarrow 1-\exp(-m1_0\cdot t^*) < \exp(-\{(m2- m2_0)-(m1-m1_0)\}\cdot t^*)\cdot\{1 - \exp(-m2_0)\cdot t^*\} \quad (2.5.10)$$

Using corollary 1.6.1, we have

$(m1-m1_0)-(m2-m2_0) \geq 0$.

This inequality implies that

$$1 \leq \exp(-\{(m2- m2_0)-(m1-m1_0)\}\cdot t^*). \quad (2.5.10a)$$

Using corollary 1.6.1 again, we have

$m1_0 \leq m2_0$ or $-m1_0 \geq -m2_0$.

This implies that

$$1-\exp(-m1_0)\cdot t^*) \leq 1 - \exp(-m2_0). \quad (2.5.10b)$$

Here, (2.5.10) is proved by "multiplying" (2.5.10a) and (2.5.10b). Hence the proof of this theorem is completed.

Similarly, we have the following theorem. The proof of this theorem is almost the same as the proof of Theorem 2.3.2 and even simpler, thus the proof is omitted.

Theorem 2.5.3:

Let $t^*(m,\alpha)$ be the solution of

$$1 - \exp(-(m-m_0+1)\cdot t^*) / (1- \exp(-t^*)) = \alpha. \quad (2.5.11)$$

If $0 < m1 < m2$, then $t^*(m1,\alpha) \leq t^*(m2,\alpha)$. Equality holds only when $m1-m1_0= m2-m2_0$.

The results of these three theorems may seem a little puzzling at first. If the burn-in lot size, n, is fixed and the more defective items are in the lot, i.e., the larger m is, then the waiting times between early failures should be smaller than the corresponding ones in the similar lot with fewer defective items. In this occasion, it's pretty natural for us to guess that $P(W_1 < t^*, W_2 < t^*, \ldots , W_{m2_0} < t^* \mid m=m2) \geq P(W_1 < t^*, W_2 < t^*, \ldots, W_{m1_0} < t^* \mid m=m1)$ for a given t^* and $m2 \geq m1$. This contradicts the result of Theorem 2.5.1. The reason for this is that m_0 is increased faster than the increase in m, or a larger portion of defectives must be screened out when the burn-in lot has more defectives in it, since n is fixed. More precisely, as shown in Lemma 1.2.3 in Procedure 0, m_0/m is an increasing function of m, a larger (smaller) portion of defectives should be eliminated through burn-in if the proportion of defectives is larger (smaller) in the burn-in lot. It takes more (less) time to screen out a larger (smaller) portion of the defectives. This is the crucial factor which makes Theorem 2.5.1 true.

Note: t^* is also an increasing function in α and ρ.

§2.6 The relation between m and t* when m/n is fixed.

For any electronic component production lot from a production line, we can assume that the ratio of the number of defectives to the number from this production lot is a constant, say r. In this section, we'll study the relation between t* and m when m/n is a constant. This is a very important subject, since it can give us the idea about the minimum amount of burn-in time t*. It is also very useful in designing the burn-in facility: how large the burn-in lot-size should be.

When the ratio of the number of the defectives in the burn-in lot over the burn-in lot size, m/n, is a constant, $t^*(m,\alpha)$ is a strictly decreasing function in m (or n) as m (or n) is increased (if some conditions are satisfied). We'll prove that $t^*(m,\alpha)$'s, the solutions of the equations (2.3.3), (2.3.4) and (2.3.5), have this property in the following three theorems.

Before stating and proving these theorems, let's define the following relation. For integers m1, m2, n1 and n2 with

$$0 < m1 < m2,\ 0 < n1 < n2,\ m1 < n1,\ m2 < n2 \text{ and } m1/n1=m2/n2=r. \qquad (2.6.1)$$

Theorem 2.6.1:

For any m1, m2, n1 and n2, let $t^*(mi,\alpha)$, i=1 or 2, be the solution of

$$\Pi_{mi-mio+1\le j\le mi}\ (1 - \exp(-(mi+j+1)\cdot t^*)) = \alpha. \qquad (2.6.2)$$

If m1, m2, n1, n2 are defined as in (2.6.1), α is in (0,1) and fixed, and for any fixed $t^* > 0$, the following condition is satisfied

$$1-\exp(-(m2-m2o+1)\cdot t^*)\cdot\{1-\exp(-m2o\cdot t^*)\}/\{1-\exp(-t^*)\}$$

$$>\{1-\exp(-(m1-m1o+1)t^*)\cdot(-\exp(-m1o\cdot t^*)/[m1o\cdot\{1-\exp(-t^*)\}]\}^{m1o}, \qquad (2.6.3)$$

then $t^*(m1, \alpha) > t^*(m2, \alpha)$.

Remark: Denote $W_{j,mi}$ be the W_j when m=mi, then (2.6.3) denotes the following probability inequality:

$$1 - P(W_{m2o,m2} > t^*) \cdot P(W_{1,m2} < t^*)/P(W_{m2,m2} < t^*)$$

$$> \{ 1 - [P(W_{m1o,m1} > t^*) \cdot P(W_{1,m1} < t^*)]/[P(W_{m1,m1} < t^*) \cdot m1o] \}^{m1o} \quad (2.6.4)$$

Proof:

Using Corollary 2.3.1, the monotone property of the left hand side of (2.6.2), to prove this theorem is the same as to prove, under the hypothesis of this theorem, for fixed

$t^* > 0$,

$$\Pi_{j=m1-m1o+1,m1} (1-\exp(-(m1-j+1) \cdot t^*))$$

$$< \Pi_{j=m2-m2o+1,m2} (1 - \exp(-(m2-j+1) \cdot t^*)). \quad (2.6.5)$$

To prove (2.6.5), we have to use the fact that arithmetic mean is greater than geometric mean.

$$\{[\Sigma_{j=m1-m1o+1,m1} (1 - \exp(-(m1-j+1) \cdot t^*))]/m1o\}^{m1o}$$

$$\geq \Pi_{j=m1-m1o+1,m1} (1 - \exp(-(m1-j+1) \cdot t^*)).$$ In addition, (2.6.6)

$$\{[\Sigma_{j=m1-m1o+1,m1} (1 - \exp(-(m1-j+1) \cdot t^*))]/m1o\}^{m1o}$$

$$= \{1 - \exp(-(m1-m1o+1) \cdot t^*) \cdot (1 - \exp(-m1o \cdot t^*))/[m1o \cdot \{1-\exp(-t^*)\}]\}^{m1o}. \quad (2.6.7)$$

Using (2.6.2), when mi=m2, we have

$$\Pi_{j=m2-m2o+1,m2} (1 - \exp(-(m2-i+1) \cdot t^*))$$

$$\geq 1 - \Sigma_{j=m2-m2o+1,m2} \exp(-(m2-i+1) \cdot t^*)$$

$$= 1 - \exp(-(m2-m2o+1) \cdot t^*) \cdot \{1-\exp(-m2o \cdot t^*) /\{1-\exp(-t^*)\}. \quad (2.6.8)$$

From (2.6.5), (2.6.6) and (2.6.7), we know that (2.6.3) is true if

$$1 - \exp(-(m2-m2o+1) \cdot t^*) \cdot \{1-\exp(-m2o \cdot t^*)\}/\{1-\exp(-t^*)\}$$

$$> \{1 - \exp(-(m1-m1o+1) \cdot t^*) \cdot (1-\exp(-m1ot^*)/[m1o \cdot \{1-\exp(-t^*)\}]\}^{m1o}. \quad (2.6.9)$$

The proof of this theorem is now complete.

Theorem 2.6.2:

Let $t^*(mi, \alpha)$, i=1or 2, be the solution of

$$1 - \{ \exp(-(mi-mi_0+1)\cdot t^*) - \exp(-(mi+1)\cdot t^*) \}/(1- \exp(-t^*)) = \alpha. \tag{2.6.10}$$

If m1, m2, n1, n2 are defined as in (2.6.10), α is in (0,1) and fixed, and for any $t^* > 0$,

$$\{1 - \exp(-m1_0\cdot t^*)\}\cdot\exp(-(m1-m1_0+1)\cdot t^*)$$
$$< (\text{ or} \geq) \{1 - \exp(-m2_0\cdot t^*)\}\cdot\exp(-(m2-m2_0+1)\cdot t^*) \tag{2.6.11}$$

then $t^*(m1, \alpha) > (\text{or} \leq) \; t^*(m2, \alpha)$.

Note:

$$(1-\exp(-m1_0\cdot t^*))\cdot\exp(-[m1-m1_0+1]\cdot t^*) > (1-\exp(-m2_0\cdot t^*))\cdot\exp(-[(m2-m2_0)+1]\cdot t^*)$$

can be denoted as

$$P(W_{m1-m1_0-1,m1} < t^*)\cdot P(W_{m1_0,m1}) > t^*)$$
$$> P(W_{m2-m2_0-1,m2} < t^*)\cdot P(W_{m2_0,m2}) > t^*)$$

Proof:

We prove this theorem by using the same idea, Corollary 2.2.1, as in proving Theorem 2.6.1. If we can prove, for fixed t^*,

$$1 - \{\exp(-(m1-m1_0+1)\cdot t^*) - \exp(-(m1+1)\cdot t^*) \}/(1- \exp(-t^*))$$
$$< 1 - \{\exp(-(m2-m2_0+1)\cdot t^*) - \exp(-(m2+1)\cdot t^*) \}/(1- \exp(-t^*)), \tag{2.6.12}$$

then we have proved this theorem. (The proof of the other case is the same.)

Inequality (2.6.12) is true.

$$\Leftrightarrow \exp(-(m1-m1_0+1)\cdot t^*)- \exp(-(m1+1)\cdot t^*)$$
$$> \exp(-(m2-m2_0+1)\cdot t^*) - \exp(-(m2+1)\cdot t^*).$$

$$\Leftrightarrow 1-\exp(-m1_0\cdot t^*)$$
$$> \exp(-(m2-m2_0+1)\cdot t^*-(m1-m1_0+1)\cdot t^*)-\exp(-(m2+1)\cdot t^*-(m1-m1_0+1)\cdot t^*).$$

$\Leftrightarrow 1-\exp(-m1_0 \cdot t^*) > \exp(-[(m2-m2_0)-(m1-m1_0)] \cdot t^*) - \exp(-((m2-m1)+m1_0) \cdot t^*)$.

$\Leftrightarrow (1-\exp(-m1_0 \cdot t^*)) \cdot \exp(-[m1-m1_0+1] \cdot t^*) > (1-\exp(-m2_0 \cdot t^*)) \cdot \exp(-[(m2-m2_0)+1] \cdot t^*)$

The proof of this theorem is now complete.

Theorem 2.6.3:

Let $t^*(mi, \alpha)$, i=1or 2, be the solution of

$$1 - \exp(-(mi-mi_0+1) \cdot t^*) / (1-\exp(-t^*)) = \alpha. \quad (2.6.14)$$

If m1, m2, n1, n2 are defined as in (2.6.1) and α is in (0,1) and fixed, then

$t^*(m1, \alpha) \geq t^*(m2, \alpha)$.

Note: this is the case of the most conservative bound of (2.2.4). The $t^*(m, \alpha)$ in this case has the desired monotonicity without any additional condition.

Proof:

As the proofs of the previous theorems: using Corollary 2.3.1, we only need to prove, for a fixed $t^* > 0$,

$$1 - \exp(-(m1-m1_0+1) \cdot t^*)/(1-\exp(-t^*))$$

$$\leq 1 - \exp(-(m2-m2_0+1) \cdot t^*)/(1-\exp(-t^*)). \quad (2.6.15)$$

Equation (2.6.15) is true

$$\Leftrightarrow \exp(-(m1-m1_0+1) \cdot t^*) \geq \exp(-(m2-m2_0+1) \cdot t^*). \quad (2.6.16)$$

From corollary 1.6.1, we know that

$$(m2-m2_0)-(m1-m1_0) \geq 0. \quad (2.6.22)$$

Hence, (2.6.14) is true. This theorem is completely proved.

Note:

When m/n is a constant, from the numerical computation, we know that $m-m_0$ is the most crucial value in calculating t^*. Moreover, from the numerical computation of t^* when (2.3.3) or (2.3.4) or (2.3.5) is used, we have the following theorem.

Theorem 2.6.4:

Let m_1 be the smallest positive integer with $m_1 0=1$. For $i=1,2,3,\cdots$

If m_{i+1} is the smallest positive integer more than a positive integer with m_i-$m_i 0+1=m_{i+1}-m_{i+1}0$, then

$t^*(m_i, \alpha) \le t^*(m_i, \alpha) \le \cdots \le t^*(m_{i+1}-1, \alpha)$ and $t^*(m_{i+1}, \alpha) < t^*(m_i, \alpha)$,(2.6.23)

when (2.3.3) or (2.3.4) or (2.3.5) is used to solve t^*.

Proof:

First, let's consider the case $(m+1)_0 = m_0+1$, i.e. $(m+1)-(m+1)_0 = m-m_0$. Using g1, g2 and g3, which are defined in Lemma 2.3.1 by letting $k = m-m_0+1$ and $n^*=m$, we know that $t^*(m,\alpha) \le t^*(m+1,\alpha)$.

For the case that $(m+1)_0 = m_0$, i.e. $(m+1)-(m+1)_0 = m-m_0+1$, we will have $t^*(m,\alpha) > t^*(m+1,\alpha)$, since

1. $\Pi_{(m+1)-(m+1)0+1,m+1}(1-\exp(-i\cdot t^*))$

   $= \Pi_{(m+1)-(m+1)0+1,m+1}(1-\exp(-i\cdot t^*))\{1-\exp(-(m+1)\cdot t^*)\}/$

   $\{1-\exp(-(m-m_0+1)\cdot t^*)\}$

   and $\{1-\exp(-(m+1)\cdot t^*)\}/\{1-\exp(-(m-m_0+1)\cdot t^*)\}$ is greater than 1.

2. $1 - \Sigma_{(m+1)-(m+1)0+1,m+1}(1-\exp(-i\cdot t^*))$

   $= 1 - \Sigma_{m-m0+1,m}(1-\exp(-i\cdot t^*)) + \{\exp(-(m-m_0+1)\cdot t^*) - \exp(-(m+1)\cdot t^*)\}$

   $\ge 1 - \Sigma_{m-m0+1,m}(1-\exp(-i\cdot t^*))$.

3. $1 - \exp\{-[(m+1)-(m+1)_0+1]\cdot t^*\}/\{1-\exp(-t^*)\}$

   $= 1 - \exp\{-[m-m_0+2]\cdot t^*\}/\{1-\exp(-t^*)\} \ge 1 - \exp\{-[m-m_0+1]\cdot t^*\}/\{1-\exp(-t^*)\}$.

So, this theorem is proved.

This theorem tells us how to pick the best (n,m) pair (to ensure the smallest possible t*) under some cost or lot size constraints. That is, let n* be the available upper bound of burn-in lot size and $m^*=[n^*\cdot r]$, the greatest integer in $n^*\cdot r$. If $m^*_0 = (m^*-1)_0$, then use $t^*(m^*,\alpha)$ as the upper bound on waiting time and n* as the lot size; otherwise, use the closest n, which is small than n*, with $t^*([n\cdot r]_0,\alpha) < t^*(m^*,\alpha)$.

Note: In some cases, we may be interested in minimizing the cost per item under test. But, here, we are interested in getting the most suitable local minimum of $t^*(m,\alpha)$.

§2.7 The value of t* - No information about m is available

Consider our reliability function $R(t; D, m, n) = 1 - \{(m-J_D)/(n-J_D)\}(1-\exp(-t))$ and the probability $P(R(t; D, m, n) \geq \rho) \geq \alpha$. What is the best t* to fit our needs? The range of m is contained in [0,n-1]. Some information about the upper bound or the lower bound of m may be available. Here, "m" may have a known prior distribution. Can we apply the available information about m to get a better t*so that, under this rule, the duration of burn-in can be shortened and $P(R(t; D, m, n) \geq \rho)$ is at least equal to the value specified? We discuss all of these in this and the following sections.

From Lemma 1.2.2, we know if m=n, burn-in does not improve reliability. Assume $m \leq n-1$. In addition, if $R(t, D=0, m, n) \geq \rho$, no burn-in is needed, too. Lemma 1.2.2 also shows that $R(t, D=0, m, n) \geq \rho$, no burn-in is required if $\rho \leq \exp(-t)$ and $m \leq n\cdot\{(1-\rho)/(1-\exp(-t))\}$. So, from now on, assume $\rho > \exp(-t)$ and $m > n\cdot\{(1-\rho)/(1-\exp(-t))\}$.

If m is close to n or no information about m is available, the safest way to have $R(t;D,m,n) \geq \rho$ is to screen out all the defectives during burn-in. This tells us to replace mo by n-1 in (2.3.3), (2.3.4) and (2.3.5). Now, we are going to solve

$$\prod_{i=1,n-1} \{1 - \exp(-i\cdot t^*)\} = \alpha . \tag{2.7.1}$$

$$1 - \{(\exp(-t^*)-\exp(-((n-1)+1)\cdot t^*)\}/(1-\exp(-t^*)) = \alpha, \text{ or} \tag{2.7.2}$$

$$1 - \exp(-t^*)/\{1-\exp(-t^*)\} = \alpha. \tag{2.7.3}$$

The binary searches algorithms used in this section to solve (2.7.1), (2.7.2) and (2.7.3) are the same as the binary search defined in §1.2. When the shortest duration of burning is required, we'll use the t* by solving (2.7.1). Otherwise,we can use the t* by solving (2.7.2) or (2.7.3). It is obvious to us that to solve (2.7.1) is much more complicated than to solve the other two equations. This is one reason why equations (2.7.2) and (2.7.3) are considered here. Another reason is that an equation like (2.7.3) had been used extensively in solving t* in Marcus & Blumenthal (1974).

Define

$m0_n^* = n-1- (1-\rho)/(\rho-\exp(-t))$ and

$m0_n$= the smallest integer more than or equal $n-1- (1-\rho)/(\rho-\exp(-t))$.

In this case, $P(R(t; D, m, n) \geq \rho) \geq \alpha$ is ensured if we have $P(R(t; D, n-1, n) \geq \rho) \geq \alpha$, that is at least $m0_n$ defectives will be eliminated through burn-in. Equations (2.3.3), (2.3.4) and (2.3.5) can be rewritten as

$$\Pi_{i=1,m0_n}\{1 - \exp(-i \bullet t^*)\} = \alpha . \qquad (2.7.4)$$

$$1 - \{(\exp(-\{(n-1) - m0_n +1\} \bullet t^*) - \exp(-\{(n-1)+1\} \bullet t^*)\}/(1-\exp(-t^*)) = \alpha. \qquad (2.7.5)$$

$$1-\exp((-\{(n-1) - m0_n +1\} \bullet t^*)/(1-\exp(-t^*)) = \alpha. \qquad (2.7.6)$$

The solutions, t*s, of (2.7.4), (2.7.5) and (2.7.6) preserve the same ordering property for each t* as the corresponding solution to each of (2.3.3), (2.3.4) and (2.3.5). This means that the solution of (2.7.4) is less than the solution of (2.7.5), and both of them are less than the solution of (2.7.6). Smaller t* implies shorter duration of burn-in. In order to save the time of burn-in, we should use the smallest t* if computation is not a problem.

From Theorem 2.5.1, Theorem 2.5.2, Theorem 2.5.3 and Lemma 1.2.3, we know that the t* obtained in this section is the most conservative one to ensure $P(R(t; D, m, n) \geq \rho) \geq \alpha$ when m=n-1 is used. In addition, the t* obtained in this section, when equation (2.7.4) or (2.7.5) is used, is smaller than the corresponding t* in Marcus and Blumenthal (1974), since they eliminated more terms in evaluating t*. We keep the lot size, n, as a parameter in calculating t*. More comparisons between their results and our accomplishments will be given in §2.14.

To derive t*, we can use the binary search algorithm specified in §2.2. For the solution of equation (2.5.6), we can also use a fixed point iterative algorithm to find t* as described in section 2.4:

A Fixed Point Iterative Algorithm to Find t* by Using (2.7.6):

1) Let $t_0 = -\{\ln(1 - \alpha)\}/(n-m0_n+1)$.
2) Let $t_i = t_0 - \{\ln(1 - \exp(-t_{i-1}))\}/(n-m0_n)$ for i=1,2,...
3) Stop when $|t_i - t_{i-1}| \leq e$, where e is a prespecified error bound.
4) Let $t^*=t_i$.

From Lemma 2.4.1, we know that the sequence $\{t_i\}$ defined by the above algorithm converges to the solution of (2.7.6). Moreover, this algorithm and a similar algorithm in the following section will be used to compare the duration of burn-in between the case that no information about m is available (m=n-1) and the case that a smaller upper bound of m is available (m<n-1).

§2.8 The value of t^* - an accurate upper bound of m is given

Suppose $m \le n \cdot \beta = m_\beta$ and $m/n = r$, then $0 \le r \le \beta \le 1$. We have

$$m_0 \sim m_0^*$$
$$= \{m \cdot (1-\exp(-t)) - n \cdot (1-\rho)\}/\{\rho - \exp(-t)\}$$
$$\le \{n \cdot \beta \cdot (1-\exp(-t)) - n \cdot (1-\rho)\}/\{\rho - \exp(-t)\} = m_{\beta 0}^* \sim m_{\beta 0}. \tag{2.8.1}$$
$$s_{m0} = m_0/m \sim \{r \cdot (1-\exp(-t))-(1-\rho)\}/\{r \cdot (\rho-\exp(-t))\} = s_{m0}^*, \tag{2.8.2}$$
$$s_{m\beta 0}^* = \{\beta \cdot (1-\exp(-t))-(1-\rho)\}/\{r \cdot (\rho-\exp(-t))\} = m_{\beta 0}^*/m. \tag{2.8.3}$$

Trivially, $m_0^* \le m_{\beta 0}^*$ and $s_{m0}^* \le s_{m\beta 0}^*$. We'll expect to eliminate a larger number and a larger portion of defectives from the burn-in lot than is truly required if an upper bound for m is used as its true value. This is similar to the result of lemma 1.2.3 and the note of corollary 1.6.1.

We know that $1 - \{(m-j)/(n-j)\} \cdot (1-\exp(-t)) \ge \rho$ if $j \ge m_0$. So, any $j \ge m_{\beta 0}$ ($\ge m_0$) will guarantee $1 - \{(m-j)/(n-j)\} \cdot (1-\exp(-t)) \ge \rho$. In additon, $P(J \ge m_{\beta 0}) \le P(J \ge m_0) = P(R(t; D, m, n) \ge \rho)$. To ensure $P(R(t; D, m, n) \ge \rho) \ge \alpha$, we can do it by finding a t^* such that $P(J \ge m_{\beta 0}) \ge \alpha$ holds when this screening procedure is used. To save the amount of burn-in time, we only need to solve for the equality case. This is the same as to find a t^* with

$$P(W_1 < t^*, W_2 < t^*, \ldots, W_{m_{\beta 0}} < t^*)$$
$$= \prod_{n \cdot \beta - m_{\beta 0}+1 \le i \le n \cdot \beta} (1 - \exp(-i \cdot t^*)) = \alpha. \tag{2.8.4}$$

We can reduce the amount of computation in finding t^* when some of the 'insignificant' terms in (2.8.4) are deleted as in (2.3.4) or (2.3.5), we have

$$1 - \{\exp(-(n \cdot \beta - m_{\beta 0}+1) \cdot t^*) - \exp(-(n \cdot \beta+1) \cdot t^*)\}/(1-\exp(-t^*)) = \alpha, \text{ or} \tag{2.8.5}$$
$$1 - \exp(-(n \cdot \beta - m_{0\beta}+1) \cdot t^*)/(1-\exp(-t^*)) = \alpha. \tag{2.8.6}$$

The value of t^* can be found by using the binary search described in §2.2. For solving (2.8.6) we can use the following fixed point iterative algorithm which is similar to the one in the previous section.

A Fixed Point Iterative Algorithm to Solve t^* by Using (2.8.6):

Let $t_0 = -\{\ln(1-\alpha)\}/(n\bullet\beta - m_{\beta 0} + 1)$.

Let $t_i = t_0 - \{\ln(1-\exp(-t_{i-1}))\}/(n\bullet\beta - m_{\beta 0}+1)$ for $i=1,2,\ldots$.

Stop when $|t_i - t_{i-1}| \le e$, where e is a prespecified error bound.

Let $t^* = t_i$.

From Lemma 2.5.2, we know that this algorithm is convergent (by letting $\mu = n\bullet\beta - m_{0\beta}+1$ and if $\alpha > 1 - \{(n\bullet\beta - m_{\beta 0}+1)/ [1+(n\bullet\beta - m_{\beta 0}+1)]\}$). It is obvious to us that the t_0 in the above algorithm is the previous t_0 in §2.7 divided by $(n-m0_n+1)/(n\bullet\beta - m_{\beta 0} + 1)$. So, we can see that the duration of burn-in is reduced if a smaller (more accurate) upper bound of m is given.

Using Theorem 2.5.1, Theorem 2.5.2 and Theorem 2.5.3, we have the following corollary to tell us how to choose t^* if a least upper bound of m is available.

Corollary 2.8.1:

If the least upper bound of m is available, then the t^* derived based on this is smaller than the t^* derived with $m=n-1$ (no information about m is available).

So, to save the amount of time in burn-in, use the t^* derived by using the least upper bound of m if it's available. In addition, for the same burn-in lot, this Corollary also shows that the t^* derived in this section is smaller than the t^* derived in the previous section.

To have a more clear idea about the amount of time which is required for burn-in, we'll study the expected duration of burn-in for the various cases in §2.12 ~ §2.14 and in §2.15.

§2.9 The value of t* - given the prior distribution of m

Let M be the number of the defective items in a randomly selected burn-in lot of size n. Assume that M is distributed as $P(M=m|\theta)$, $m=0,1,2,\cdots,n$, where θ can be a vector parameter. In the previous sections, we let $t^*(m,\alpha)$ be the t* obtained by solving (2.3.3) or (2.3.4) or (2.3.5) when the value of M is m. In this section, we are trying to determine what an appropriate t* could be if the distribution of M, $P(M=m|\theta)$, is known.

The Bayes approach for us to solve this problem is to find a t* such that

$$
\begin{aligned}
&P(R(t;\, D,\, M,\, n)\geq\rho) \\
&= E_M\{P(R(t;\, D,\, M,\, n)\geq\rho) \mid \theta\} \\
&= \sum_{m=1,n} P(R(t;\, D,\, M,\, n)\geq\rho|M=m)\cdot P(M=m|\theta) \\
&= \sum_{n\cdot\{(1-\rho)/(1-\exp(-t))\}< m\leq n} P(J\geq m0 \mid M=m)\cdot P(M=m|\theta) + P(M\leq n\cdot\{(1-\rho)/(1-\exp(-t))\}|\theta) \\
&= \sum_{n\{(1-\rho)/(1-\exp(-t))\}< m\leq n} \{\prod_{i=m-m0+1,m} [1-\exp(-i\cdot t^*)]\}\cdot P(M=m|\theta) + P(M\leq n\cdot\{(1-\rho)/(1-\exp(-t))\}|\theta) \\
&\geq \alpha. \qquad (2.9.1)
\end{aligned}
$$

The summation of m is started from $n\cdot\{(1-\rho)/(1-\exp(-t))\}$, since $R(t;\, D,\, m,\, n)$ is always greater than or equal to ρ if no burn-in is required, i.e. $m < n\cdot(1-\rho)/(1-\exp(-t))$.

As we replace (2.3.2) by (2.3.4) or (2.3.5) in calculating t*, we are throwing away some insignificant terms to simplify the computation and getting an upper bound on the true t*. Using (2.3.4) and (2.3.5) in (2.9.1) we have

$P(R(t;\ D, M, n) \geq \rho)$

$$= \sum_{n\{(1-\rho)/(1-\exp(-t))\} < m \leq n} \{\prod_{i=m-m0+1,m} [1-\exp(-i \cdot t^*)]\} \cdot P(M=m|\theta) + P(M \leq n\{(1-\rho)/(1-\exp(-t))\}|\theta)$$

$$\geq \Sigma_{m=0,n}\{1 - [\exp(-(m-m_0+1) \cdot t^*) - \exp(-m+1) \cdot t^*)]/[1-\exp(-t^*)]\} \cdot P(M=m)$$

$$= 1 - [\exp(-t^*)/(1-\exp(-t^*))] \cdot \{\Sigma_{m=0,n}[\exp(-(m-m_0) \cdot t^*) - \exp(-m) \cdot t^*)] \cdot P(M=m) \quad (2.9.2)$$

We approximate the right side of (2.9.2) by replacing the integer $(m-m_0)$ with the non-integer

$m-m_0^*$

$$= m - [m \cdot (1-\exp(-t)) - n \cdot (1-\rho)/(\rho-\exp(-t)) = -m \cdot (1-\rho)/(\rho-\exp(-t)) + n \cdot (1-\rho)/(\rho-\exp(-t))$$

$$= -(c \cdot m + b), \text{ where } c = (1-\rho)/(\rho-\exp(-t)) \text{ and } b = -n \cdot c. \quad (2.9.3)$$

Using (2.9.3) and letting $MGF_M(x)$ be the moment generating function of M, we get from (2.9.2)

$P(R(t;\ D, M, n) \geq \rho)$

$$\geq 1 - [\exp(-t^*)/(1-\exp(-t^*))] \cdot \{\exp(b \cdot t) \cdot MGF_M(c \cdot t^*) - MGF_M(-t^*)\} \quad (2.9.4)$$

$$\geq 1 - [\exp((b-1) \cdot t^*)/(1-\exp(-t^*))] \cdot MGF_M(c \cdot t^*) \quad (2.9.5)$$

As mentioned in §2.3, setting expressions (2.9.1) or (2.9.4) or (2.9.5) equal to α, we can solve for t^* by a binary search. Using lemma 2.3.1, we know that they are all monotonically increasing functions in t^*. Let $x=\exp(-t^*)$ and $f(x)$=the left-hand side of (2.9.1) (or (2.9.4) or (2.9.5)), then the binary search described in Section 2.3 will give us the right t^* up to any level of precision. However, the computation to get t^* is more complicated than the previous two cases when (2.9.1) is used. To avoid this kind of much longer computation, let's consider some possible alternatives rather than using (2.9.4) or (2.9.5).

We may screen out all the defectives when ρ is very close to 1 or M is close to n as we did in §2.7, if we ignore the prior distribution of m. We might think about using $t^* = \sum t^*(m)P(M=m|\theta) = E(t^*(M))$ and using numerical computation to justify how it works, but this doesn't clearly express the relation between t* and $P(R(t; D, M, n)\geq\rho)\geq\alpha$. Let's try to use the $100\cdot\beta$, where $\beta=\alpha^{1/2}$, percentile of M, $m^\beta = \min\{m: P(M\leq m)\geq\beta\}$, as our true m and then apply (2.3.3) (or (2.3.4) or (2.3.5)) to calculate $t^*(m^\beta,\beta)$. This seems to be a reasonable approach, since $P(M \leq m^\beta) \geq \beta > \alpha$. In this case

$$\begin{aligned}
&P(R(t; D, M, n)\geq\rho)\\
&= P(R(t; D, M, n)\geq\rho| M \leq m^\beta)\cdot P(M \leq m^\beta)+ P(R(t; D, M, n)\geq\rho|M \geq m^\beta)\cdot P(M\geq m^\beta)\\
&\geq \beta\cdot P(M \leq m^\beta) + P(R(t; D, M, n)\geq\rho|M \geq m^\beta)\cdot P(M \geq m^\beta)\\
&\geq \beta^2 + P(R(t; D, m, n)\geq\rho|M \geq m^\beta)\cdot P(M \geq m^\beta) \geq \beta^2 = \alpha. \qquad (2.9.6)
\end{aligned}$$

Here, $P(R(t; D, m, n)\geq\rho) \geq \alpha$ is guaranteed, and $t^*(m^\beta,\alpha)$ is easier to calculate, so we could use this t* as a substitute. (Note: $P(R(t; D, m, n)\geq\rho|M \geq m^\beta)\cdot P(M \geq m^\beta) \leq P(M \geq m^\beta) \leq 1-\sqrt{\alpha}$ which should be small.)

Usually, we assume that M has a binomial or a Poisson distribution. If an accurate t* is required to reduce the duration of burn-in, it would be better to solve (2.9.1) directly by using the binary search algorithm as before. Otherwise, (2.9.4) or (2.9.5) or (2.9.6) can be used to find the appropriate t*. The moment generating function, MGF_b, of a binomial distribution function, $P(M=m|\phi)$, is $MGF_b(s) = (\phi\cdot\exp(s) + 1 - \phi)^n$. And, the moment generating, MGF_p, of a poisson distribution function, $P(M=m|\phi)$, is $MGF_p(s) = \exp(\phi\cdot(\exp(s) - 1)$. If M has a binomial distribution, then the MGF is an increasing functions in s, so we can use the binary search to find t* without any hesitation.

Note: A numerical table used to compare the results in this section, when M is binomial, will be given in Appendix I.

§2.10 Probability of stopping this screening procedure before the j+1st failure of a defective item is observed after the failure of the j-th defective item.

The results in this and the following sections are independent of the equation used to find t^* in the previous sections, so the t^* here can be any t^* (or $t^*(m,\alpha)$) in the previous sections. Most of the results in this and the next section can be found in Marcus and Blumenthal (1974). For the sake of completeness, their results are listed here.

Define $P(J \geq j|m,t^*)$ as the probability of the number (random) of the failed defectives up to the time of stopping equals or exceeds j when this sequential screening scheme with t^* is used, and the true number of defectives put on burn-in is m. We have, as (2.3.2) or (2.3.3), for j = 1, 2, ..., m,

$$P(J \geq j|m,t^*)$$

$$= P(W_1 < t^*, W_2 < t^*, ..., W_j < t^* |m)$$

$$= \prod_{i=m-j+1,m} (1-\exp(-i \cdot t^*)) \quad \text{for } j=1,2,...,m \tag{2.10.1}$$

and $P(J \geq 0|m,t^*) = 1$. In addition, for j = 1, 2, ... ,

$$P(J = j |m,t^*)$$

$$= P(J \geq j | m,t^*) - P(J \geq j+1|m,t^*)$$

$$= \prod_{i=m-j+1,m} (1-\exp(-i \cdot t^*)) - \prod_{i=m-j,m} (1-\exp(-i \cdot t^*))$$

$$= \prod_{i=m-j+1,m} (1-\exp(-i \cdot t^*)) (1 - (1-\exp(-(m-j) \cdot t^*))$$

$$= \exp(-(m-j) \cdot t^*) \prod_{i=m-j+1,m} (1-\exp(-i \cdot t^*)) \tag{2.10.2}$$

and $P(J = 0|m,t^*) = \exp(-m \cdot t^*)$.

§2.11 The expected duration of burn-in.

Clearly, from (2.3.1), the expected ith waiting time between failure $E(W_i|m) = 1/(m+1-i)$, where m is the number of defective items. Denote the expected i^{th} waiting time between failures given that it does not exceed t* by $E(W_i \mid W_i \leq t^*, m)$ as in Marcus and Blumenthal(1974). We have

$$E(W_i \mid W_i \leq t^*,m)$$

$$= \int_{0 \leq s < t^*} \{(m+1-i)\cdot s\}\cdot \exp(-(m+1-i)\cdot s)/(1-\exp(-(m+1-i)\cdot t^*))ds$$

$$= \{1/(1-\exp(-(m+1-i)\cdot t^*))\}\cdot\{-t^*\cdot\exp(-(m+1-i)\cdot t^*) + \int_{0 \leq s < t^*} \exp(-(m+1-i)\cdot s)ds$$

$$= \{1/(1-\exp(-(m+1-i)\cdot t^*))\}\cdot\{-t^*\cdot\exp(-(m+1-i)\cdot t^*) + [1- \exp(-(m+1-i)\cdot t^*)]/(m+1-i)$$

$$= \{1/(m+1-i\} - \{ t^*\cdot\exp(-(m+1-i)\cdot t^*)/(1-\exp(-(m+1-i)\cdot t^*))$$

$$= E(W_i) - t^*/[\exp((m+1-i)\cdot t^*)-1]. \qquad (2.11.1)$$

Let's denote the random duration of burn-in as D. For a given m and t*, we have, as in Marcus and Blumenthal (1974),

$$E(D| m,t^*)$$

$$= t^* + \sum_{j=0,m} P(J = j) \sum_{i=0,j} E(W_i \mid W_i < t^*,m), \qquad (2.11.2)$$

where we let $E(W_0) = E(W_0 \mid W_0 < t^*,m) = 0$.

When the assumed m is more than its true value, the t* based on this value is larger than what it should be. In this case, $E(D|m,t^*)$ is longer than what it should be, or we will have a longer expected duration.

If the number of defectives, M, has prior distribution $P(M=m|\theta)$, then

$E(D|t^*)$

$$= \sum_{m=0,n} P(M=m|\theta) \cdot E(D|m,t^*)$$

$$= t^* + \sum_{m=0,n} P(M=m|\theta) \sum_{j=0,m} P(J=j\backslash m) \cdot \sum_{i=0,j} E(W_i|W_i < t^*,m) \qquad (2.11.3)$$

Equation (2.11.2) or (2.11.3) tells us that a minimum amount of burn-in time, t^*, is required, if this stopping rule is used. In addition, this is obvious from Stopping Rule (S.2.1).

§2.12 $E(D|m,t^*)$ is an increasing function in t^* when m (n, ρ and α) is fixed.

The main theorem of this section shows that $E(D|m,t^*)$ is a strictly increasing function in t^* for fixed m and fixed n (the size of burn-in lot). This theorem can be used to compare the difference between two expected durations, for a given burn-in lot, if two different estimates of m are used in obtaining t^*. Define

$$g(m,t^*) = E(D|m,t^*) = t^* + \sum_{j=0,m} P(J=j) \cdot \sum_{i=0,j} E(W_i \mid W_i < t^*, m), \qquad (2.12.1)$$

we have the following theorem.

Theorem 2.12.1

Given fixed m and n, for t^* in $(0,\infty)$, $g(m,t^*)$ is a strictly increasing function in t^*.

Proof:

$g(m,t^*) = t^* + \sum_{j=0,m} P(J=j \mid m,t^*) \cdot \sum_{i=0,j} E(W_i \mid W_i < t^*,m)$

$= t^* + \sum_{j=1,m} \exp(-(m-j)\cdot t^*) \prod_{i=m-j+1,m} (1-\exp(-i\cdot t^*)) \cdot \sum_{i=1,j} \{1/(m+1-i) - t^*/[\exp((m+1-i)\cdot t^*) - 1]\}$, since $W_0=0$.

The following lemmas and corollaries show that, for fixed m,

1) For $j=0,1,\cdots,m$, $P(J=j \mid m,t^*)$ skews to the right, as t^* is increased. This fact is getting more significant as j approaches m.
2) For $i=0,1,\cdots,m$, $E(W_i \mid W_i < t^*,m)$ is a strictly increasing function in t^*.
3) For $j=0,1,\cdots,m$, $\sum_{i=0,j} E(W_i \mid W_i < t^*,m)$ is a strictly increasing function in t^*, too.
4) For $j=0,1,\cdots,m$, $\sum_{j=0,m} P(J=j \mid m,t^*) \cdot \sum_{i=0,j} E(W_i \mid W_i < t^*,m)$ is increased as t^* increases.

So, we conclude that $g(m,t^*)$ is a strictly increasing function in t^*.

Lemma 2.12.1

Given a fixed m, for t>0, u>0 and j is a nonnegative integer less than or equal to m, define $f(t,u,j,m)=P(J=j|t,m)/P(J=j|u,m)$.

If t>u, then f(t,u,j,m) is a strictly increasing function in j. Thus, $P(J=j \mid t^*,m)$ skews to the large j as t* is increased.

Proof:

For j=0,1,...,m-1,

$$f(t,u,j,m)$$

$$=\{\exp(-(m-j)\cdot t)\cdot\prod_{i=m-j+1,m}(1-\exp(-i\cdot t))\}/\{\exp(-(m-j)\cdot u)\cdot\prod_{i=m-j+1,m}(1-\exp(-i\cdot u))\}.$$

$$=\exp(-(m-j)\cdot(t-u))\cdot\prod_{i=m-j+1,m}\{(1-\exp(-i\cdot t))/(1-\exp(-i\cdot u))\}.$$

$$f(t,u,j+1,m)=\exp(-(m-j-1)\cdot(t-u))\cdot\prod_{i=m-j,m}\{(1-\exp(-i\cdot t))/(1-\exp(-i\cdot u))\}$$

$$=\exp(t-u)\cdot\{(1-\exp(-(m-j)\cdot t)/(1-\exp(-(m-j)\cdot u)\}\cdot\prod_{i=m-j+1,m}\{(1-\exp(-i\cdot t))/(1-\exp(-i\cdot u))\}$$

$$=\exp(t-u)\cdot\{(1-\exp(-(m-j)\cdot t)/(1-\exp(-(m-j)\cdot u)\}\cdot f(t,u,j,m).$$

$$f(t,u,j+1,m) > f(t,u,j,m)$$

$$\iff \exp(t-u)\cdot\{(1-\exp(-(m-j)\cdot t)/(1-\exp(-(m-j)\cdot u)\} > 1$$

$$\iff \exp(t)\cdot(1-\exp(-(m-j)\cdot t) > \exp(u)\cdot(1-\exp(-(m-j)\cdot u)$$

$$\iff t > u.$$

The proof of Lemma 2.12.1 is completed.

Lemma 2.12.2

Given a fixed m, for i=0,1,...,m, $E(W_i \mid W_i < t^*,m)$ is a strictly increasing function in t*.

Proof:

$$E(W_i \mid W_i < t^*,m) = 1/(m+1-i) - t^*/(\exp((m+1-i)\cdot t^*)-1).$$

Define $h(t) = t/(\exp(a\cdot t)-1)$.

The following proposition shows that $h(t)$, $h(t)=t/\{\exp(a \bullet t)-1\}$, is a strictly decreasing function in t. Let $t^*=t$ and $m+1-i=a$ in $h(t)$, Lemma 2.12.2 is proved.

Proposition:

For t in $(0,\infty)$ and any $a>0$, $h(t) = t/(\exp(a \bullet t)-1)$ is a strictly decreasing function in t.

Proof:

For t in $(0,\infty)$,

$h'(t) = \{(\exp(a \bullet t)-1)-t \bullet a \bullet \exp(a \bullet t)\}/(\exp(a \bullet t)-1)^2$.

$h'(t) < 0$, is true,

$\Leftrightarrow \{(\exp(a \bullet t)-1)-t \bullet a \bullet \exp(a \bullet t)\}/(\exp(a \bullet t)-1)^2 < 0$

$\Leftrightarrow \exp(a \bullet t)-1-t \bullet a \bullet \exp(a \bullet t) < 0$

$\Leftrightarrow 1-\exp(-a \bullet t)-a \bullet t < 0$.

Define $h1(t) = 1-\exp(-a \bullet t)-a \bullet t$. We have $h(0)=0$ and $h1(0)=0$.

If we can prove that $h1(t)$ is a strictly decreasing function in t for t in $(0,\infty)$, we have $h1(t)<0$ since $h1(0)=0$. For t in $(0,\infty)$, $h1(t)<0$, implies that $h'(t)<0$. So, for t in $(0,\infty)$, $h(t)$ is a strictly decreasing function in t.

$h1'(t) = a \bullet \exp(-a \bullet t)-a=a \bullet (\exp(-t)-1) < 0$ if $t > 0$.

Hence, $h1(t)$ is a strictly decreasing function in t for $t > 0$.

Corollary 2.12.1

For t^* in $(0,\infty)$, $\Sigma_{i=0,j}E(W_i \mid W_i < t^*,m)$ is a strictly increasing function in t^*.

Proof:

A linear combination of strictly increasing functions with positive coefficient is strictly increasing.

To prove that $\Sigma_{j=0,m}P(\ J = j\ |m,t^*)\bullet\Sigma_{i=0,j}E(\ W_i \mid W_i < t^*,m)$ is a strictly increasing function in t^*, we need to prove the following lemma.

Lemma 2.12.3

For i=0,1,...,m, assume a(i), b(i), c(i) and d(i) are nonegative real numbers with

1) $\Sigma_{0,m}\ a(i)=1$,
2) $\Sigma_{0,m}\ b(i)=1$,
3) b(i)/a(i) is increased as i is increased,
4) c(i) and d(i) are monotonically increasing in i,
5) For each i, d(i) > c(i).

Under the above condition, $\Sigma_{i=0,m}a(i)\bullet c(i) < \Sigma_{i=0,m}b(i)\bullet d(i)$.

Proof:

Let $i_L=\max_{i=0,1,\ldots,m}\{i;\ a(i)\geq b(i)\}$ and $i_U=\min_{i=0,1,\ldots,m}\{i;a(i)\leq b(i)\}$. (*)

It is clear that $i_L \leq i_U$. Moreover, $i_L = i_U$ or $i_L + 1 = i_U$.

Let $d_L=d(i_L)$, $d_U = d(i_U)$ and $\underline{d}=(\ d_L+d_U\ )/2$.

Using 4), we have $d(0)<d(1)<\bullet\bullet\bullet< d_L \leq \underline{d} \leq d_U< d(i_{U+1})<\bullet\bullet\bullet<d(m)$. (**)

Finally, we have

$\Sigma_{i=0,m}\{a(i)\bullet c(i)-b(i)\bullet d(i)\}$

$< \Sigma_{i=0,m}\{a(i)\bullet d(i)-b(i)\bullet d(i)\}$

$< \Sigma_{i=0,m}d(i)\bullet\{a(i)-b(i)\}$

$< \Sigma_{i=0,m}\underline{d}\bullet\{a(i)-b(i)\}$, by using (*) and (**),

$= 0$.

So, $\Sigma_{i=0,m}a(i)\bullet c(i) < \Sigma_{i=0,m}b(i)\bullet d(i)$. The proof of this lemma is now complete.

Corollary 2.12.2

For fixed m and n, $\Sigma_{j=0,m}P(\ J = j\ )\bullet\Sigma_{i=0,j}E(\ W_i \mid W_i < t^*,m)$ is a strictly increasing function in t^*.

Proof:

For $t2^*>t1^*$ and $j=0,1,2,\cdots,m$, define

$a(j) = P(J=j|m,t1^*)$,

$b(j) = P(J=j|m,t2^*)$,

$c(j) = \Sigma_{i=1,j} E(W_i|W_i<t1^*,m)$.

$d(j) = \Sigma_{i=1,j} E(W_i|W_i<t2^*,m)$.

It is trivial that $\Sigma_{j=0,m} a(j) = \Sigma_{j=0,m} b(j) = 1$.

From Corollary 2.12.1, we have $d(j) > c(j)$ for $j=0,1,\cdots,m$ and both of them are strictly increasing function in j.

In addition, Lemma 2.12.1 shows that $b(j)/a(j)$ is increased as j is increased.

Hence, all the conditions of the above lemma are satisfied, the proof of this corollary is completed.

Using the above lemmas and corollaries, we have proved Theorem 2.12.1.

In a given burn-in lot, with fixed n, ρ and α, this theorem shows that the expected duration of burn-in is longer if the used t^* is larger. In addition, for fixed n, ρ and α, we know, in §2.3, that $t^*(m,\alpha)$, is a monotonically increasing function in m, so the expected duration is longer if the used estimated m is larger. The following corollary summarizes this result.

Corollary 2.12.3:

For a given burn-in lot with unknown number of defectives, m, if $m1 < m2$ be two estimates of m, in addition to having

(1) $t^*(m1,\alpha) < t^*(m2,\alpha)$, we have

(2) $E(D| m,t^*(m1,\alpha)) < E(D|m,t^*(m2,\alpha))$

where $t^*(m,\alpha)$ can be the solution of equation (2.3.3) or (2.3.4) or (2.3.5).

If m is overestimated, we have a conservative rule: $P(R(t; D,m, n)\geq\rho)$ is larger than what is truly required and longer expected duration of burn-in. If m is underestimated we might not be able to achieve our desired reliability goal.

Suppose that m has prior distribution $P(M=m|\varnothing)$, for $m=0,1,\cdots,n$. In (2.11.3), we have $E(D|t^*)= \sum_{m=0,n} P(M=m|\varnothing)\bullet\ E(D|m,t^*(\alpha))$ where $t^*(\alpha)$ is the solution of equation (2.9.1) or (2.9.4) or (2.9.5). Using the above results, we have the following corollary.

Corollary 2.12.4:

If $t2^* > t1^*$, then $E(D|t1^*) < E(D|t2^*)$.

§2.13 The expected duration of burn-in for different lot size, n, when the ratio m/n is a (fixed) constant.

This case, m/n is a constant, say r, is a very important case in studying this screening procedure, since we usually assume that the ratio of the number of defectives to the burn-in lot size, from a production line is a constant. The expected duration gives us the idea about how long this burn-in will take if this screening procedure is used. Some special (m,n) pairs may give us shorter expected duration of burn-in just as happened in Theorem 2.6.1. For economic purposes (or any other purposes), we need to know what these (m,n) pairs are.

In this case, as $t^*(m,\alpha)$, $E(D|m,\alpha)$ is not a monotonic function in m. This can be seen from the numerical computation. However, $E(D|m,\alpha)$ does preserve the similar property as $t^*(m,\alpha)$ has, that is the following conjecture.

Conjecture 2.13.1:

Let m_1 be the smallest positive integer with $m_1 0=1$. For $i=1,2,\cdots$, if m_{i+1} is the smallest positive integer more than the positive integer m_i with $m_{i+1}-m_{i+1}0 = m_i-m_i0+1$, then

$$E(D|m_i,\alpha)\le E(D|m_i+1,\alpha)\le\cdots\le E(D|m_{i+1}-1,\alpha) \text{ and } E(D|m_{i+1},\alpha) < E(D|m_i,\alpha), \quad (2.13.1)$$

when (2.3.3.) or (2.3.4) or (2.3.5) is used. And, let n_i be the smallest positive integer more than or equal m_i/r.

This Conjecture tells us to find the sequence $\{m_i\}$, i=1,2,•••, as defined before. In addition, we use n_i, which is the smallest integer more than or equal m_i/r, as the corresponding size of burn-in lot. Then, for this sequence of number pairs (m_i,n_i), we are expecting to have

$$E(D|m_{i+1},\alpha) < E(D|m_i,\alpha) \text{ for } i = 1,2, \cdots. \qquad (2.13.2)$$

And, $\{n_i\}$, i=1,2,•••, are our corresponding burn-in lot sizes. In order to achieve the best (economic) result through burn-in, we choose the appropriate n_i from this sequence.

Note: The above discussion is for the case that the burn-in lot size is to be determined. If the burn-in lot size is fixed in advance, then the average of the expected burn-in times, $E(D|m,\alpha)/n$, would be a good criterion.

§2.14 The difference between the results of Marcus and Blumenthal (1974) and the results of this procedure

Marcus and Blumenthal (1974) considered the case that no information about m is available and solved our problem in different set up. Eventually, they try to find a t*, as we did here, to ensure that the number of defectives left after burn-in is bounded with at least some desired level of probability. They were very successful in finding the appropriate t* and ruled out the effect of m. The information about m was assumed unavailable in their paper and they didn't consider the other cases as we discussed here. Mainly, they solved t* by using equation (2.3.5) and had the solved t* not depend on m and n with the cost of a longer duration of burn-in.

Here, we suggest to use equation (2.3.3) and to use the available information about m to solve the desired t*. The duration of burn-in is shorter but it depends on the information about m. When no information about m is available, we can use n-1 as the upper bound of 'i' in (2.3.3) and ç (defined in §2.2) as the corresponding lower bound. In this case, we'll still have the solved t* less than the t* in their paper.

Two numerical algorithms are suggested here for the calculation of t*. The binary search algorithm is good for all equations (2.3.3), (2.3.4), (2.3.5) and its extensions. The fixed point iterative algorithm is used for solving equation (2.3.5) which was used to construct the tables in their paper. These two algorithms are easy to write into a computer code and they are proved to be convergent.

Since the information about m was assumed unknown in their paper, they didn't have any discussion based on this. Here, we have put a lot of effort in investigating the

properties of $t^*(m,\alpha)$ and $E(D|m,t^*)$ based on the available information about m and intend to find a best burn-in lot size to fulfill our goal under the possible constraints, like cost and time limitations, imposed on burn-in.

CHAPTER III

PROCEDURE II - SMALL SAMPLE THEORY

§3.1 Idea of Procedure II

The maximum likelihood estimator of m (Johnson 1962) is $m^{ml}(D) = J_D/F(D,$ where F is the cumulative distribution of the lifetimes of the defectives. The maximum likelihood estimator of m is used to estimate R(t;D,m,n) and the maximum likelihood estimator $R(t;D,m^{ml}(D),n)$ is obtained. Consider the stopping rule defined by stopping as soon as $R(t;D,m^{ml},n) \geq k(\rho,\alpha,m,n)$. We attempt to find k such that $P(R(t; D, m, n) \geq \rho) \geq \alpha$ can be achieved. Here $k=k(\rho,\alpha,m,n)$ is a function of m . This is not what we'd like to see because the value of m is not given exactly, but, we'll try to determine an appropriate $k = k(\rho,\alpha,m,n)$ such that it is not very sensitive to m. We also hope that a more appropriate k can be derived if more information about m is available. Another interesting aspect of this screening procedure, as with the previous two procedures, is the procedure's expected duration. We'll see that this is a function of k. The expected duration of burn-in is also a function of m, since k is a function of m.

The small sample theory for all of these will be studied in this chapter. The corresponding large sample theory will be studied in the next chapter. The following is a brief summary of this chapter. The definition of the stopping rule developed in this chapter is given in Section 3.2. In addition, a generalization of this stopping rule is also given in this section. The probability that this rule stops after exactly j defective items have been removed from the burn-in lot is obtained in Section 3.3. In addition, the probability for the generalized stopping rule is derived in §3.4. How k is obtained, based on small sample theory, is discussed in §3.5, §3.6 and §3.7. In most of the last part of this chapter, the performance of this stopping rule based on the expected duration criterion is discussed. The number of the defective items left after this burn-in procedure is given in Section 3.11.

§3.2 The Stopping Rule

Let D = the duration of burn-in and J_D = the number of failed defective items observed up until time D

$R(t; D, m, n) = 1 - ((m - J_D)/(n - J_D))\bullet(1 - \exp(-t)).$ (3.2.1)

If m is replaced by its maximum likelihood estimator, $m^{ml} = J_D/(1 - \exp(-D))$, then we have an estimator of R(t; D, m, n),

$R(t; D, m^{ml}, n)$

$= 1- ((m^{ml} - J_D)/(n - J_D))\bullet(1 - \exp(-t))$

$= 1- (((J_D/(1 - \exp(-D)) - J_D)/(n - J_D))\bullet(1 - \exp(-t))$

$= 1- ((J_D/(n - J_D))\bullet(\exp(-D)/(1 - \exp(-D))\bullet(1 - \exp(-t))$

$= 1- ((J_D/(n - J_D))\bullet(1/(\exp(D) - 1))\bullet(1 - \exp(-t)).$ (3.2.2)

The idea of the stopping rule to be studied is:

Stop as soon as $R(t;D,m^{ml},n) \geq k = k(\rho,\alpha,m,n)$.

The value of k will have to be determined, and since $R(t; 0, m^{ml}, n) = 1$, a minimum burn-in period will be required to ensure $P(R(t; D, m, n) \geq \rho) \geq \alpha$. The rule will be presented formally below in a manner which facilitates the study of its properties.

Based on (3.2.2), we reformulate the stopping rule in terms of a sequence of possible stopping times $\{S_j=s(n,j,k), j=0, 1, 2, \ldots ,n\}$ as follows. For a given k, $0 \leq k \leq 1$,

$R(t; D, m^{ml}, n) \geq k$

$\Leftrightarrow$ $1- (J_D/(n - J_D))\bullet(1/(\exp(D)-1)\bullet(1 - \exp(-t)) \geq k$ (3.2.3)

$\Leftrightarrow$ $1 - k \geq (J_D/(n - J_D))\bullet\{1/(\exp(D)-1\}\bullet(1 - \exp(-t))$

$\Leftrightarrow$ $\exp(D) -1 \geq (J_D/(n - J_D))\bullet\{(1 - \exp(-t))/(1 - k)\}$

$<=>$ $D \geq \ln\{[(J_D/(n - J_D)]\bullet[(1 - \exp(-t))/(1 - k)] + 1\}$ (3.2.4)

Let $S_j = s(n, j, k) = \ln[(j/(n - j))\bullet((1 - \exp(-t))/(1 - k)) + 1]$ for $j=0, 1, 2, \ldots, n-1$;

$S_n = s(n, n, k) = \infty.$ (3.2.5)

Note:

$S_0 = 0$ and Inequality (3.2.4) is always true at $D=0$.

The sequence of possible stopping times, $\{S_j, j=0, 1, 2, \ldots, n\}$, is a sequence of fixed values. Here, D, the time to failure of the defective item is random and unknown, and the index J of S_J at which the rule stops is random.

Inequality (3.2.4) gives us a way to re-express the stopping rule: given that j failures have been observed, if the failure time of the (j+1)st defective item exceeds the right hand side of (3.2.4) (i.e., after observing j failures, no additional failure has been observed by time S_j), then stop burn-in; otherwise continue burn-in. Formally, we have the **stopping rule**:

Stop burn-in at S_{j-1} when the first j is reached with $T_j > S_{j-1}$ for j = 2, 3, ..., n. (S.3.0)

In this rule, $k = k(\rho,m,n)$ needs to be chosen such that $P(R(t; D, m, n) \geq \rho) \geq \alpha$ is guaranteed. Note: the value of j starts from two. If j starts from one, $T_1 > S_0 = 0$ is always true, i.e., this rule always stops at time 0.

Discussion:

1. The function s(n,j,k) is a monotonic increasing function in j. This means that the more failures are observed, the longer the burn-in must be run. It also means that you don't stop when a failure occurs, but you stop when a gap between failures becomes large.

2. The function $s(n, j, k)$ is a monotonic increasing function in k. This tells us that to get a higher (estimated) reliability a longer burn-in duration is required.
3. The function $s(n, 0, k) = 0$. This would tell us to stop immediately if there is no failure at time zero, and we ignore this requirement.
4. The function $s(n, 1, k) > 0$. We use this as the minimum duration of burn-in.
5. The function $s(n, n, k)=\infty$, and $s(n, j, k) < \infty$ for $j=0, 1, \ldots, n-1$. If all the items under burn-in are defectives, then burn-in cannot achieve the reliability goal. If $m \leq n-1$, then burn-in will be terminated at one of $s(n, j, k), j=1, 2, \ldots, m$.

Note:

1. In this stopping rule, we require that the index of j be at least one. If we did not, since the MLE of $R(t;D,m,n)$ is unity up to the time of the first failure, the burn-in would be stopped immediately unless there was a failure at start-up.
2. If $T_1 > S_1$, then stop burning-in. In this case, we have $T_2 > T_1 > S_1$.
3. The upper bound of m is n-1, so n is used as the upper bound of j.
4. If S_{j-1} is replaced by S_j in (S.3.0), a more conservative rule is obtained. The reason is that $S_j > S_{j-1}$ and the probability of T_j not exceeding S_j is higher than the probability that T_j does not exceed S_{j-1}.

As a consequence of the fourth note above, we can replace S_{j-1} by S_j in this stopping rule. The advantage of doing this is: a more conservative rule is obtained and the requirement that at least one failure must be observed, imposed by using the MLE, is removed. So, our new **stopping rule** is

Stop burn-in at S_j when the first j is reached with $T_j > S_j$
for $j = 1, 2, \ldots, n$. (S.3.1)

The probability integral transformation converts an exponentially distributed random variable T over $(0,\infty)$ into a uniformly distributed random variable U over (0,1). This transformation will enable us to analyze this stopping rule (S.3.1) through a sequence of ordered random observations $U_1=1-\exp(-T_1)$, $U_2=1-\exp(-T_2)$, •••, $U_j=1-\exp(-T_j)$, ••• from uniform(0,1), where T_i's are the ordered (random) lifetimes of the failed defectives under burn-in.

Let's define

$$
\begin{aligned}
C_j &= c(n, j, k) \\
&= 1-\exp(-s(n, j, k))=1-1/\{1+[(1-\exp(-t))/(1-k)]\cdot(j/(n-j)\} \\
&= (j\cdot(1-\exp(-t))/[(1-k)\cdot(n-j)+j\cdot(1-\exp(-t))], \qquad (3.2.4)
\end{aligned}
$$

where c(n, j, k) is a monotonic transformation of s(n, j, k). Based on this we have our transformed **stopping rule** :

Stop burn-in at c(n, j, k) when the first j is reached with U_j > c(n, j, k), for j=1, 2, •••, n. (S.3.2)

Here, j starts from 1 rather than 2.

We should be very cautious when stopping rule (S.3.1) or (S.3.2) is in use. One very essential requirement for this stopping rule is that j must be at least 1. This is due to the application of the MLE of m. More precisely, let's look at (3.2.3). We can see that (3.2.3) is always true if $J_D=0$ or if there is no need of burn-in for this production lot at the very beginning of our screening procedure. However, we believe that the desired reliability goal cannot be achieved or the proportion of defectives in this production is higher than acceptable without burning-in. In order to clarify this problem, we need that either a minimum duration of burn-in is required or at least one failed defective should be observed before burn-in is stopped if we decided to use burn-in for a given production lot. Moreover, we noted before, that if S_j is increased as j is increased, a much more conservative rule can be obtained if S_j is replaced by S_{j+i} for any positive integer i. The choice of i is based on our reliability consideration. To consolidate all the points discussed here, we have the

Modified General Stopping Rule.

For an appropriate choice of a positive integer i, stop burning-in at S_{1+i} if $T_1 > S_{1+i}$.

For j=2,3,... , stop burning-in at S_{j+i-1} when the first j is reached with $T_j > S_{j+i-1}$. (S.3.3)

or

if $T_1 > S_{1+i}$, then stop at S_{1+i}.

For j=2,3, ... , stop burning-in at S_{j+i} when the first j is reached with $T_j > S_{j+i}$. (S.3.4)

Here, (S.3.3) corresponds to (S.3.0) and (S.3.4) corresponds to (S.3.1). These two rules clearly indicate that this screening procedure will not be stopped until some minimum number of failures, i+1, have been observed.

First of all, we'll consider this procedure by assuming that m is known. Later on, we'll briefly discuss how k depends on the available information about m. Since k is the only unspecified crucial parameter of this stopping rule, if we know k, then we know this stopping rule.

§3.3 $P(J=j|m,k)$ of the Original Stopping Rule (S.3.0) or (S.3.2)

Stopping rule (S.3.2) says: Stop burn-in at the first j with $T_j > S_j$ where T_j is the life time of the jth failed defective item, and stop burn-in at S_j, for j=1, 2, •••, m+1. (To get Stopping Rule (S.3.0), replace S_j by S_{j-1} for j=2,•••,n.) So the probability that this burn-in procedure stops at time S_j is $P(J=j-1|m,k)$. This probability can be calculated when m is known and k is given; i.e.,

$P($ stop burn-in at $S_{j+1} | m,k)$

$= P($ j failed defective items observed during burn-in when burn-in is stopped $| m,k)$

$= P(J=j|m,k)$ for $j=0,1,2,\ldots,m$. (3.3.1)

Let's evaluate (3.3.1). For j=0,

$P(J=0|m,k)$

$= P($ burn-in is stopped before the occurrence of any failure $| m,k)$

$= P(T_1 > S_1 | m,k)$

$= P(U_1 > C_1 | m,k)$

$= (1-C_1)^m$ (3.3.2)

For $j=1,2,\ldots,m$,

$P(J=j|m,k)$

$= P($ j failed defective items have been observed when burn-in is stopped $| m,k)$

$= P($ burn-in is stopped at $S_{j+1} | m,k)$

$= P(T_1 \le S_1, T_2 \le S_2, \ldots, T_j \le S_j, T_{j+1} > S_{j+1} | m,k)$

$= P(U_1 \le C_1, U_2 \le C_2, \ldots, U_j \le C_j, U_{j+1} > C_{j+1} | m,k)$

$$= \{m!/(m-j-1)!\} \cdot \int_{0<u_1<C_1} \cdots \int_{u_{j-1}<u_j<C_j} \int_{C_{j+1}<u_{j+1}<1} (1-u_{j+1})^{m-j-1} du_{j+1} du_j \ldots du_1$$

$$= \{m!/(1!(m-j)!)\} \cdot (1-C_{j+1})^{m-j} \int_{0<u_1<C_1} \int_{u_1<u_2<C_2} \cdots \int_{u_{j-1}<u_j<C_j} du_j \ldots du_2 du_1 \quad (3.3.3)$$

To evaluate the integral in (3.3.3), define

$$B_0 = 1. \quad (3.3.4)$$

$$B_1 = \int_{0<u_1<C_1} du_1 = C_1 = C_1 \cdot B_0 \quad (3.3.5)$$

$$B_2 = \int_{0<u_1<C_1} \int_{u_1<u_2<C_2} du_2 du_1$$

$$= \int_{0<u_1<C_1} (C_2-u_1)\, du_1$$

$$= C_2 \cdot \int_{0<u_1<C_1} du_1 - \int_{0<u_1<C_1} u_1 du_1$$

$$= C_2 \cdot B_1 - (1/2) \cdot (C_1)^2 \quad (3.3.6)$$

$$B_3 = \int_{0<u_1<C_1} \int_{u_1<u_2<C_2} \int_{u_2<u_3<C_3} du_3 du_2 du_1$$

$$= \int_{0<u_1<C_1} \int_{u_1<u_2<C_2} (C_3-u_2)\, du_2 du_1$$

$$= C_3 \cdot B_2 - \int_{0<u_1<C_1} \int_{u_1<u_2<C_2} u_2 \, du_2 du_1$$

$$= C_3 \cdot B_2 - (1/2) \cdot \int_{0<u_1<C_1} [(C_2)^2 - (u_1)^2]\, du_1$$

$$= C_3 \cdot B_2 - (1/2) \cdot (C_2)^2 \cdot B_1 + (1/2) \cdot (1/3) \cdot (C_1)^3$$

$$= \sum_{i=1,3} (-1)^{i-1} \cdot (1/i!) \cdot (C_{3-i+1})^i \cdot B_{3-i} \quad (3.3.7)$$

$$\vdots$$

By mathematical induction, it can be shown that

$$B_j = \int_{0<u_1<C_1} \int_{u_1<u_2<C_2} \cdots \int_{u_{j-1}<u_j<C_j} du_j \ldots du_2 du_1$$

$$= \sum_{i=1,j} (-1)^{i-1} \cdot (1/i!) \cdot (C_{j-i+1})^{i} \cdot B_{j-i} \qquad \text{for } j=1,2,\ldots,m. \qquad (3.3.8)$$

From (3.3.3) and (3.3.8), we have

$$P(J=j|m,k) = (m!/(m-j)!) \cdot (1-C_{j+1})^{m-j} \cdot B_j \qquad \text{for } j=0,1,2,\ldots,m. \qquad (3.3.9)$$

Summarizing the above results, we have:

Theorem 3.3.1

$$P(J=j|m,k)= (m!/(m-j)!) \cdot (1-C_{j+1})^{m-j} \cdot B_j \qquad \text{for } j=0,1,2,\ldots,m. \qquad (3.3.10)$$

where j denote the total number of failed defective items observed when burn-in is stopped at time S_{j+1} (Note: $C_j = 1 - \exp(-S_j)$).

§3.4 P(J=j|m,k) of the Modified General Rules (S.3.3) (or (S.3.4))

As in the previous sections, we define P(J=j|m,k) as the probability that there are j failed defectives observed when this screening procedure is stopped. This is the probability, for an appropriate i, that $T_1 \le S_{1+i}$ (or $T_1 \le S_{1+i}$), $T_2 \le S_{2-1+i}$(or $T_2 \le S_{2+i}$), ..., $T_j \le S_{j-1+i}$ (or $T_j \le S_{j+i}$) and $T_{j+1} > S_{j+1-1+i}$ (or $T_{j+1} > S_{j+1-i}$); or $U_1 \le C_{1+i}$ (or $U_1 \le C_{1+i}$), $U_2 \le C_{2-1+i}$ (or $U_2 \le C_{2+i}$), ..., $U_j \le C_{j-1+i}$ (or $U_j \le C_{j+i}$) and $U_{j+1} > C_{j+1-1+i}$ (or $U_{j+1} > C_{j+1+i}$) given m and k. Starting from here, the following derivation will be based on stopping rule (S 3.3) (For stopping rule (S.3.4), P(J=j|m,k) can be derived in the same way by replacing the corresponding C_j (or S_j) in the following derivations). The derivation of P(J=j| m,k) for these two generalized stopping rules is similar to the derivation of P(J=j| m,k) in the previous section, except some differences in the integration part of this probability. For the sake of completeness, it is given here. The next part of this section is the derivation of P(J=j|m,k) for j=0,1,2,•••,m.

For j=0,

P(J=0| m,k)

= P(burn-in stopped before the occurrence of any failure | m,k)

= $P(T_1 > S_{1+i} \mid m,k)$

= $P(U_1 > C_{1+i} \mid m,k)$. (3.4.1)

For j=1, 2, ..., m,

P(J=j|m,k)

= P(j failed defective items observed when burn-in is stopped | m,k)

$= P(\text{ burn-in is stopped at time } S_{j-1+i} \mid m,k)$

$= P(T_1 \le S_{1+i}, T_2 \le S_{2-1+i}, \dots, T_j \le S_{j-1+i}, T_{j+1} > S_{j+1-1+i} \mid m,k)$

$$= P(U_1 \le C_{1+i}, U_2 \le C_{2-1+i}, \dots, U_j \le C_{j-1+i}, U_{j+1} > C_{j+1-1+i} \mid m,k) \tag{3.4.2}$$

Let's evaluate (3.4.1) and (3.4.2). It is clear that

$$P(J=0|m,k) = P(U_1 > C_{1+i}) = (1-C_{1+i})^m \tag{3.4.3}$$

$$P(J=1|m,k) = P(U_1 \le C_{1+i}, U > C_{2-1+i} |m,k) = m\cdot C_{1+i}\cdot(1-C_{2-1+i})^{(m-1)}. \tag{3.4.4}$$

To evaluate $P(J=j|m,k)$ for $j=2,3, \dots, m$, let's define:

$$A_j = \int_{0<u_2<C_{2-1+i}} u_2 \int_{u_2<u_3<C_{3-1+i}} \dots \int_{u_{j-1}<u_j<C_{j-1+i}} du_j \dots du_3 du_2, \tag{3.4.5}$$

with $A_1 = C_{2-1+i}$ and $A_0 = 1$.

We have:

$$A_2 = \int_{0<u_2<C_{2-1+i}} u_2\, du_2 = (1/2)\cdot C_{2-1+i}^2 \tag{3.4.6}$$

$$A_3 = \int_{0<u_2<C_{2-1+i}} u_2 \int_{u_2<u_3<C_{3-1+i}} du_3 du_2$$

$$= C_{3-1+i}\cdot \int_{0<u_2<C_{2-1+i}} u_2\, du_2 - \int_{0<u_2<C_{2-1+i}} u_2^2\, du_2$$

$$= C_{3-1+i}\cdot A_2 - (1/3)\cdot C_{2-1+i}^3 \tag{3.4.7}$$

For $j=4,5, \dots,m$, by induction, We have

$$A_j = \int_{0<u_2<C_{2-1+i}} u_2 \int_{u_2<u_3<C_{3-1+i}} \cdots \int_{u_{j-1}<u_j<C_{j-1+i}} du_{j-1} \dots du_3 du_2$$

$$= C_{j-1+i}\cdot A_{j-1} - (1/2)\cdot\{(C_{j-1+i-1})^2 A_{j-2} - (1/3)\cdot\{(C_{j-1+i-2})^3\cdot A_{j-3} - (1/4)\cdot$$

$$\{\cdots - (1/h)\cdot\{(C_{m-h-1+i+1})^i\cdot A_{m-i} - 1/(h+1)\cdot\{\cdots - 1/(j-1+i-2)\cdot \{(C_{3-1+i})^{j-2}$$

$$\cdot A_2 - (1/j)\cdot(C_{2-1+i})^j\}\cdots\}\cdots\}\cdots\}\} \tag{3.4.8}$$

Finally, we have:

$$P(J=2|m,k) = \{m!/(1!(m-2)!)\}\bullet(1-C_{3-1+i})^{m-2}\bullet A_2 \tag{3.4.9}$$

$$P(J=3|m,k) = \{m!/(1!(m-3)!)\}\bullet(1-C_{4-1+i})^{m-3}\bullet A_3 \tag{3.4.10}$$

For j=4, 5, ... m,

$P(J=j|m,k)$

$= P(U_1\le C_{1+i}, U_{2-1+i}\le C_2, ..., U_j \le C_{j-1+i}, U_{j+1} > C_{j+1-1+i} \mid m,k)$

$= \{m!/((2-1)!\bullet(m-j-1)!)\}\bullet\int_{0<u_2<C_{2-1+i}} u_2 \int_{u_2<u_3<C_{3-1+i}} \dots \int_{u_{j-1}<u_j<C_{j-1+i}} \int_{C_{j+1-1+i}<u_{j+1}<1} (1-u_{j+1})^{m-j-1}$

$du_j \dots du_3 du_2$

$$= \{m!/(1!\bullet(m-j)!)\}\bullet(1-C_{j+1-1+i})^{m-j}\bullet A_j \tag{3.4.11}$$

Note:

$P(J=m|m,k)$

$= P(U_1\le C_{1+i}, U_2\le C_{2-1+i}, ..., U_j \le C_{j-1+i}, U_{m+1}> C_{m+1-1+i} \mid m,k)$

$= P(U_1\le C_{2-1+i}, U_2\le C_{2-1+i}, ..., U_m \le C_{m-1+i} \mid m,k)$

$= (m!)\bullet A_m.$

Hence we have the following theorem.

Theorem 3.4.1 (under the Modified Stopping Rule)

$P(J=0|m,k) = (1- {}^*C_{j+1})^m.$

$$P(J=j|m,k) = \{m!/(m-j)!\}\bullet(1- {}^*C_{j+1})^{m-j}\bullet A_j, \text{ for } j=1,2,...,m \tag{3.4.12}$$

where, for using stopping rule (S.3.3)

${}^*C_j = C_{j-1+i}$ for $j=2,3,...,m$ and

${}^*C_0 = {}^*C_1 = C_{1+i}.$

for using stopping rule (S.3.4)

$^{*}C_j = C_{j+i}$ for j=0,1,2,3,...m

Equations (3.3.10) of Theorem 3.3.1 and (3.4.12) of Theorem 3.4.1 have identical expresions. If the C's and B's in equation (3.3.10) are replaced by the *C's and A's, then we have equation (3.4.12). Hereafter, we'll use

$P(J=j|m,k) = (m!/(m-j)!)(1-C_{j+1})^{m-j} B_j$ for j=0,1,2,...,m (3.4.13)

to denote these probabilities. It is trivial that the result which is derived based on (3.4.13) should be good for both (3.3.10) and (3.4.12). This means that the probability for stopping rules (S.3.1), (S.3.2), (S.3.3), (S.3.4) have similar expressions, except for minor changes in the definition of C_{j+1} and B_j.

§3.5 The determination of k when m is given.

For a given m, we have defined $m_0^* = (m\cdot(1-\exp(-t))-n\cdot(1-\rho))/(\rho-\exp(-t))$ and m_0 = the smallest integer more than or equal to m_0^*, and know that $R(t; D, m, n) \geq \rho$ if and only if $j \geq m_0$. In this screening procedure, k is chosen such that

$P(R(t; D, m, n) \geq \rho \mid \text{stopping rule}) \geq \alpha$.

In §3.8, we'll discuss the determination of k based on the available information about m. Let's denote this in the following inequality.

$$\sum_{j=m_0,\, m} P(J=j|m,k) \;\geq\; \alpha \tag{3.5.1}$$

or

$$\sum_{j=m_0,\, m} (m!/(m-j)!)\cdot(1-C_{j+1})^{m-j}\cdot B_j \;\geq\; \alpha\,. \tag{3.5.2}$$

Any k that satisfies (3.5.2) will give us $P(\, R(t; D, m\,, n) \geq \rho\,) \geq \alpha$ when this sequential stopping rule is used. The possible stopping time of this screening procedure will be prolonged if k is increased, since $C_j = j\cdot(1-\exp(-t))/[(n-j)\cdot(1-k)+j\cdot(1-\exp(-t))]$ is an increasing function of k. In order to reduce the duration of burn-in, we'd like to find the smallest k with $P(R(t; D, m, n) \geq \rho) \geq \alpha$. This k can be obtained by solving

$$\sum_{j=m0,\, m} (m!/(m-j)!)\cdot(1-C_{j+1})^{m-j}\cdot B_j \;=\; \alpha. \tag{3.5.3}$$

Note: We can denote the solution of (3.5.3) as $k(m,\alpha,\rho,n,t)$, since C_j is a function of r, n and 1-exp(-t).

Moreover, from the definition of B_j (equation (3.3.4) to (3.3.8) or equation (3.4.5) to (3.4.8)), we know that the upper bounds of the integrals defining B_j's are increased as k increases. So, the left hand side of equation (3.5.3) is increased as k increases. Hence, there is at most one k which can be the solution of equation (3.5.3). Eventually, for any α in [0,1] and any k_o in [0,1] with

$$\sum_{j=m0,m} P(J=j|m,k=k_o) \leq \alpha \leq \sum_{j=m0,m} P(J=j|m,k=1)=1, \tag{3.5.4}$$

there is one and only one k in $[k_o,1]$ such that

$$\sum_{j=m0,m} P(J=j|m,k) = \alpha . \tag{3.5.5}$$

So we have the following theorem.

Theorem 3.5.1

If (3.5.4) holds for a given k_o in (0,1), then, under any version of this stopping rule, the equation

$$\sum_{j=m0,m} P(J=j|m,k) = \alpha \tag{3.5.6}$$

has exactly one k as its solution in $[k_o,1]$. In addition, if

$$\sum_{j=m0,m} P(J=j|m,k=0) > \alpha \tag{3.5.7}$$

then

$$\sum_{j=m0,m} P(J=j|m,k) > \alpha \tag{3.5.8}$$

for any k in [0,1].

Note: If (3.5.7) is true, then no burn-in is necessary.

Finally, we have the following corollary.

Corollary 3.5.1:

For any α in [0,1], there is a (unique) smallest k in[0,1] such that equation (3.5.8) holds for m_0=1, or 2, ..., or m.

Note: From §3.3 and §3.5, we know that the expression for $P(J \geq m_0)$ is quite complicated. To compute k from the results of these two sections are not easy when m is moderately large. In the next chapter, possible k's will be derived through large sample theory.

§3.6 Numerical calculation of k when m is given.

For m_0 in $\{1,2, ..., m\}$, let's solve k for

$$\sum_{j=m0,\ m} P(J=j|m,k) = \alpha . \tag{3.6.1}$$

We know that the left hand side of equation (3.6.1) is a monotonically increasing function in k. If

$$\sum_{j=m0,\ m} P(J=j|m,k=0) > \alpha , \tag{3.6.2}$$

then use k=0. If

$$\sum_{j=m0,\ m} P(J=j|m,k=0) < \alpha < 1, \tag{3.6.3}$$

then use a binary search to find k as a solution of equation (3.6.1). From Theorem 3.5.1, we know that such a solution exists and is unique.

A Binary Search Algorithm for k, a solution of equation (3.5.10):

1) Let $k_1 = 1/2$

2) If $\sum_{j=m0,\ m} P(J=j|m,k_i) > \alpha$, then $k_{i+1} = k_i - (1/2)^{i+1}$. (3.6.4)

   If $\sum_{j=m0,\ m} P(J=j|m,k_i) < \alpha$, then $k_{i+1} = k_i + (1/2)^{i+1}$. (3.6.5)

3) If $i < 30$, then go to 2).

After 30 iterations, we will have $|k_{30} - k| < 10^{-9}$.

Note: Since m is unknown, an assumed value of m is used for this algorithm.

§3.7 The determination of k with the available information about m when n is fixed.

If no upper bound of m can be specified, the most conservative choice of m^e, an estimate of m, is n-1 when burn-in is required. In this case, for the corresponding k value, we may try to derive it by replacing m with n-1. This is the same as what we did in Procedure I. Similarly, if the upper bound of m is given, say $n \cdot r^e$ and r^e in [0,1], then we may hope to use $m^e = n \cdot r^e$ and to find the corresponding $m^e 0$ and k. To guarantee that a more conservative rule is obtained when a larger estimate of m is used, we need to show that $P(J \geq m^e 0 \mid m^e, k)$ is a monotonic function in m^e. However, we have the following difficulty:

Difficulty:

We should use $P(J \geq m^e 0 \mid m^e, k) = \alpha$ to solve for k. However, I cannot prove that $k(m^e)$, the solution of $P(J \geq m^e 0 \mid m^e 0, k) = \alpha$, is a monotone function of m. That is I can't see when a larger k and a larger probability $P(R(t; D, m, n) \geq \rho)$ will be obtained. One conservative k is obtained by letting $k = \max\{k(m); m$ ranges over its possible values$\}$. This k will give us $P(R(t; D, m, n) \geq \rho) \geq \alpha$ based on the following theorems.

Define $f(m^e,k) = P(J \geq m^e 0 \mid m,k)$.

Theorem 3.7.1:

For fixed n, m and m^e, $f(m^e,k)$ is a monotonic increasing function in k.

Proof:

We have $P(J \geq m^{e}{}_{0} \mid m,k) = P(U_1 < C_1, \cdots, U_{m^e} < C_{m^e} \mid m,k)$ and for $i=0,1,\cdots,n-1$, $C_i=i\cdot(1-\exp(-t))/\{(n-i)\cdot(1-k)+(i\cdot(1-\exp(-t))\}$ is a monotonically increasing function in k. Using the property of uniform order statistics, for fixed m and m^e, It's clear to see that $f(m^e,k)$ is a monotonically increasing function in k. The proof of this theorem is completed.

Note:

1. This theorem tells us if a larger k is used for a given burn-in lot then a higher reliability will be achieved through burn-in, since the boundary of stopping time is increased. That is a conservative rule is used if a larger k is used.
2. For fixed n, m, and k, $f(m^e,k)$ is a monotonic increasing function in m^e where m^e is an estimate of m.
3. For a given burn-in lot with unknown m, suppose $m1^e$ and $m2^e$ be two estimates of m with $m1^e < m2^e$. If k1 is the solution of $P(J \geq m1^{e}{}_{0} \mid m,k) = \alpha$ and k2 is the solution of $P(J \geq m2^{e}{}_{0} \mid m,k) = \alpha$, then $k1 \leq k2$.

If there is a prior distribution of m, $M \sim P(M=m|\theta)$ for $m = 0, 1, \cdots, n$, then we may use Bayes Rule or follow an argument similar to that of Procedure I to obtain an appropriate percentile of m and use the result in §3.5 and §3.6 to get the appropriate value of k.

For finding the Bayes rule, suppose $M \sim P(M=m|\theta)$ for $m = 0, 1, \cdots, n$, then

$P(R(t; D, M, n) \geq \rho) \geq \alpha$

$\Leftrightarrow \sum_{m=0,n} P(J \geq m_0|m,k)\cdot P(M=m|\theta) \geq \alpha$

$\Leftrightarrow \sum_{0\leq m\leq n\cdot\{(1-\rho)/(1-\exp(-t))\}} P(M=m|\theta)+\sum_{n\cdot\{(1-\rho)/(1-\exp(t))\}<m\leq n} P(J\geq m_0|m,k)\cdot P(M=m|\theta)$

$\geq \alpha$. (3.7.1)

Note: If $m \leq n \cdot \{(1-\rho)/(1-\exp(-t)\}$, no burn-in is required, then $P(J \geq m_0|m,k) = 1$.

To determine k using an appropriate percentile of m, let $\beta = \sqrt{\alpha}$ and $m_\beta = \min\{ m: P(M \leq m) \geq \beta\}$. Let k be the solution of k_β such that

$P(J \geq m_{\beta 0} | m_\beta) = P(R(t; D, m_\beta, n) \geq \rho) = \alpha$, where $m_{\beta 0}$ is defined as the m_0 of m. In particular, we have

$P(R(t; D, M, n) \geq \rho)$

$= \sum_{m=0,n} P(J \geq m_0|m,k_\beta) \cdot P(M=m|\theta)$

$= \sum_{0 \leq m \leq m_\beta} P(J \geq m_0|m,k_\beta) \cdot P(M=m|\theta) + \sum_{m_\beta < m \leq n} P(J \geq m_0|m,k_\beta) \cdot P(M=m|\theta)$. (3.7.2)

If the solution of k, k(m), obtained by using $P(J \geq m_0|m,k) = \alpha$ is an increasing function of m, then we have the following inequality, for $P(R(t; D, M, n) \geq \rho)$. This is the difficulty described in Section 3.6.

$P(R(t; D, M, n) \geq \rho)$

$\geq \sum_{0 \leq m \leq m_\beta} \beta \cdot P(M=m|\theta) + \sum_{m_\beta < m \leq n} P(J \geq m_0|m,k_\beta) \cdot P(M=m|\theta)$

$\geq \beta \cdot \beta + \sum_{m_\beta < m \leq n} P(J \geq m_0|m,k_\beta) \cdot P(M=m|\theta) \geq \alpha$. (3.7.3)

(Note: We are expecting to have $\sum_{m_\beta < m \leq n} P(J \geq m_0|m,k_\beta) \cdot P(M=m|\theta) \leq (1-\alpha)^{1/2}$.)

Here, the derivation of k_β is much more difficult if we are going to solve (3.7.2). However, a lot computational effort can be saved if (3.7.3) is used. In this case, we only need to find m_β and we have a conservative stopping rule. Note that, in this case, this rule might be too conservative.

§3.8 E(D|m,k) and the Distribution of D

The expected durations of burn-in based on the rules defined in this chapter (or equation (3.4.13)) is denoted by E(D|m,k) when m and k are given. From the definition of this stopping rule, it is known that burn-in will be stopped at S_1 or S_2 or ... or S_{m+1} with probability P(J=0|m,k), P(J=1| m,k), ..., P(J=m|m,k). Hence

$$E(D|m,k)$$
$$= \sum_{j=0,m} S_{j+1} \bullet P(J=j|m,k)$$
$$= \sum_{j=0,m} S_{j+1} \bullet (m!/(m-j)!) \bullet (1-C_{j+1})^{m-j} \bullet B_j \qquad (3.8.1)$$

where $C_j = 1 - \exp(-S_j)$ for j=0,1,...,m and B_j is defined as in (3.4.13).

If m has some prior distribution, M ~ P(M=m|ø), then

$$E(D|ø,k)$$
$$= \sum_{m=0,n} P(M=m|ø) \bullet E(D|m,k)$$
$$= \sum_{m=0,n} P(M=m|ø) \bullet \sum_{j=0,m} S_j \bullet (m!/(m-j)!) \bullet (1-C_{j+1})^{m-j} \bullet B_j. \qquad (3.8.2)$$

The duration of burn-in, D, of this rule has discrete values: C_i, for i=1,•••,m+1 (or n). It is clear that

$$P(D=C_{i+1}|m) = P(J_D=i) \text{ for } i=0,1,\cdots,m. \qquad (3.8.3)$$

Stopping at C_{i+1} is the same as screening out i defectives.

§3.9 To compare the duration of burn-in for a given burn-in lot when different estimates of m are used.

For a given burn-in lot with unknown m, suppose m1 and m2 are two of its estimates with $m1 < m2$, let k1 be the solution of $P(J \geq m1o \mid m1,k) = \alpha$ and let k2 be the solution of $P(J \geq m2o \mid m2,k) = \alpha$. In this section, we are interested in whether $E(D|m,k1)$ is less than $E(D|m,k2)$ or not. This can be used to decide if a larger or smaller estimate of m should be used in deriving k. In order to save burn-in time, we'd like to pick up the estimate which gives us shorter expected burn-in duration, provided the reliability goal is met.

From Section 3.2, we know that if $k1 \leq k2$, then the stopping time boundaries $C_i(k1) \leq C_i(k2))$ for $i=1,2,\cdots,n-1$, where $C_i(k1)$ is the C_i obtained by replacing k with k1 and $C_i(k2)$ is the C_i obtained by replacing k with k2. So, we have the following Theorem 3.9.1. Before proving this theorem, we need the following trivial lemma.

Lemma 3.9.1:

Suppose m1 and m2 be two positive integers with $m1 < m2$, P1 and P2 be two probability functions defined on $\{0,1,2,\cdots\}$ with $\sum_{0\leq j\leq m1}P1(j) = 1$ and $\sum_{0\leq j\leq m2}P2(j) = 1$ and, for $i=0,1,\cdots$, $\sum_{i\leq j\leq m1}P1(j) \leq \sum_{i\leq j\leq m2}P2(j)$. Let g1(i) and g2(i) be two positive and increasing functions defined on $\{0,1,2,\cdots\}$ with $g1(i) \leq g2(i)$. If the above conditions are true, then

$$\sum_{i\leq j\leq m1} g1(i)\cdot P1(j) \leq \sum_{i\leq j\leq m2} g2(i)\cdot P2(j) \qquad (3.9.1)$$

Note:

If $g1(i) = g2(i)$ for all i, we have equality in (3.9.1).

If there is at least one i with $g1(i) < g2(i)$, then we have inequality in (3.9.1).

Theorem 3.9.1:

E(D|m,k) is an increasing function in k.

Proof:

For j=2,•••,n-1, we have P(J ≥ j|m,k) = P(U_i < C_i(k) for i = 1,•••,j-1) = $\sum_{j\leq i\leq m+1}$P(J=i|m,k) is an increasing function in k. In addition, we already have that, for i=0,1,•••,n-1, C_i(k) is an increasing function in k. For 0 ≤ k1 ≤ k2 ≤1, using the above lemma, let P1(j)= P(J = j|m,k1), p2(j)= P(J = j|m,k2), g1(j)= C_j(k1) and g2(j)= C_j(k2). We have that E(D|m,k) is an increasing function in k. The proof of this theorem is completed.

Note:

Smaller estimates of m with k will give us shorter expected duration of burn-in. But, we should be very careful in his case, our reliability goal may not be able to achieved if m is under estimated.

§3.10 To compare the expected durations of burn-in, in the same burn-in facility, for the burn-in lots from different production lines.

In this section, we consider the case that two burn-in lots from two different production lines are tested in the same burn-in facility, same lot size n. To simplify the discussion of this, we assume that the defective items in these two lots have the same failure time distributions. The only difference between these two lots is the numbers of defectives in them, say m1 and m2 with m1 < m2, both of these two numbers are unknown. Let m1e and m2e be the estimates of m1 and m2, respectively, with m1e < m2e. Here, we are interested in which burn-in lot has longer expected burn-in duration.

First, let's consider the case in which the same sequence of stopping times, the same k, is used for these two lots. This is the following theorem.

Theorem 3.10.1:

For two positive integers m1 < m2, $E(D|m1,k) \le E(D|m2,k)$. (3.10.1)

Proof:

As in Theorem 3.9.1, the same k implies that we'll have the same C_i's.

We have, for $0 < m1 < m2$ and $j=0,1,2\cdots,n-1$,

$P(\ U_{i,m1} < C_i,\ \text{for } i=1,2,\cdots,j) \le P(\ U_{i,m2} < C_i,\ \text{for } i=1,2,\cdots,j)$.

where $U_{i,m1}$'s are the uniform order statistics of size m1 and $U_{i,m2}$'s are the uniform order statistics of size m2. This inequality tells us that $P(\ J_{m1} \ge j) \le P(\ J_{m2} \ge j)$ for $j=0,1,2,\cdots$, where J_{m1} is the observed number of the failed defectives if the number of defectives in burn-in is m1, J_{m2} is defined similarly. Using Lemma 3.9.1, the proof of this theorem is completed.

This theorem tells us that burn-in will be longer for the lot with more defective items. In addition, the comparison between E(D|m1e,k(m1e)) and E(D|m2e,k(m2e)) is considered, but no good results are obtained. One difficulty is that numerical results show that k(m) is not a monotone function of m.

§3.11 The Number of Defective Items Left after Burn-In

Define L as the (random) number of defective items left after burn-in and ç be any (fixed) non-negative integer. As in Marcus and Blumenthal(1974), we might want to have $P(L \le ç) \ge \alpha$ if this screening procedure is used. It is clear that $L = m - J$. We have $P(J \ge m_0 \mid m,k) \ge \alpha$, which implies that $P(L \le m-m_0 \mid m,k) \ge \alpha$. To find $P(L \le ç)$, we may try to have appropriate parameters, t and ρ, such that $m-m_0=ç$. However, we don't know what m is, even though it is a fixed integer. If the least upper bound of m (or an appropriate estimate), $n \cdot r$ with r in [0,1], is known and an appropriate k value is obtained as described in the last several sections, then

$P(L \le ç)$

$\ge P(L \le m-m_0 \mid m,k)$

$\ge P(L \le m - (n \cdot r \cdot (1-\exp(-t)) - n \cdot (1-\rho)) / (\rho - \exp(-t))) \ge \alpha.$

This inequality is true since

$m - m_0^*$

$= m - \{m \cdot (1-\exp(-t)) - n \cdot (1-\rho))/(\rho - \exp(-t))\}$

$= m \cdot \{1 - (1-\exp(-t))/(\rho - \exp(-t))\} + n \cdot (1-\rho)/(\rho - \exp(-t))$

$$= - m \cdot (1-\rho)/(\rho - \exp(-t))\} + n \cdot (1-\rho)/(\rho - \exp(-t)) \qquad (3.11.1)$$

and (3.11.1) is monotonically decreasing in m (as we have in the previous two chapters) and $n \cdot r$ is an upper bound of m.

If m has known prior distribution, say $M \sim P(M=m|\theta)$ for $m=0,1,2,\cdots n$, then

$P\{L \le ç \mid \theta,k\} = P(J \ge M-ç \mid \theta,k\}$

$= \sum_{m=0,n} P(M=m|\theta) \cdot P(J \ge m-ç \mid M=m,k)$

$$= \sum_{0 \le m \le ç} P(M=m|\theta) + \sum_{ç \le m \le n} P(M=m|\theta) \cdot P(J \ge m-ç|M=m,k) = \alpha. \qquad (3.11.2)$$

Theorem 3.11.1:

The smallest k in [0,1] such that (3.11.2) holds if $0 < \alpha < 1$ is quaranteed unique.

Proof:

Since, for m=0,1,2,•••,n, Theorem 3.7.2 tells us that $P(J \geq m\text{-}ç|M=m,k)$ is an increasing function in k. For k=1, the left hand side of (3.11.2) is

$\sum_{0 \leq m \leq ç} P(M=m|\theta) + \sum_{ç \leq m \leq n} P(M=m|\theta) \cdot P(J \geq m\text{-}ç|M=m,1)$

$= \sum_{0 \leq m \leq ç} P(M=m|\theta) + \sum_{ç \leq m \leq n} P(M=m|\theta) \cdot 1 = 1.$

For k = 0, the left hand side of (3.11.2) is

$\sum_{0 \leq m \leq ç} P(M=m|\theta) + \sum_{ç \leq m \leq n} P(M=m|\theta) \cdot P(J \geq m\text{-}ç|M=m,0).$ (*)

If (*) $\geq \alpha$ then k = 0.

If (*) $< \alpha <$ 1, then, using the property that $P(J \geq m\text{-}ç|M=m,k)$ is an increasing function in k, we have a unique k in [0,1] as the solution of (3.11.2).

The proof of this theorem is completed.

Note: $P\{ L \leq ç \mid \theta,k \}$ for some fixed ç is the probability of the number of defectives in a randomly choosen production lot after burning-in is less than ç, where the burn-in lot size is n and the number of defectives in it before burn-in has prior distribution $P(M=m|\theta)$ for m=0,...,n.

For $M \sim P(M=m|\theta)$, the expected number of defective items left after burn-in can be calculated, too. Let's consider the case that M is binomially distributed with parameter n and r where r is in [0,1]. We have

$P(M=m|r) = (n!/(m!(n\text{-}m)!)) \cdot r^m \cdot (1\text{-}r)^{n\text{-}m}$ for m=0,1,2, ... ,n (3.11.3)

and

$P(J=j,M=m|r,k)$

$= P(J=j|m,k) \cdot P(M=m|r)$

$= (m!/(m-j)!) \cdot (1-C_{j+1})^{m-j} \cdot B_j \cdot (n!/(m!(n-m)!))) \cdot r^m \cdot (1-r)^{n-m}$ (from Theorem 3.4.1)

$$= (n!/[(m-j)! \cdot (n-m)!)]) \cdot (1-r)^{n-m} \cdot (1-Cj+1)^{m-j} \cdot r^m \cdot B_j \quad (3.11.4)$$

In this case, for m = j,j+1, ... , n, the posterior probability of M given J=j is

$P(M=m \mid J=j, r, k)$

$$= P(J=j,M=m|r,k)/(\sum_{\mu=j,n} P(J=j,M=\mu|r,k)) \quad (3.11.5)$$

$= (n!/[(m-j)! \cdot (n-m)!)]) \cdot (1-r)^{n-m} \cdot (1-C_{j+1})^{m-j} \cdot r^m B_j / \{[(n!/(n-j)!)r^j \cdot \sum_{\mu=j,n}[(n-j)!]/[(\mu-j)!$
$(n-\mu)!]] \cdot (1-r)^{n-\mu} \cdot (1-Cj+1)^{\mu-j} \cdot r^{\mu-j} \cdot B_j \}$

$= \{(n-j)!/((n-m)! \cdot (m-j)!) \} \cdot \{(1-r)^{n-m} \cdot (r \cdot (1-C_{j+1}))^{m-j}/[1-r+r \cdot (1-C_{j+1})]^{n-j} \}$

$$= \{(n-j)!/((n-m)! \cdot (m-j)!)\} \cdot \{(1-r)/[(1-r+r \cdot (1-C_{j+1}))]\}^{n-m}\} \cdot \{r \cdot (1-C_{j+1})/[1-r+r \cdot (1-C_{j+1})]\}^{m-j} \quad (3.11.6)$$

which is binomial.

We have L = M-J, the number of defective items left after burn-in. So the posterior probability of L given J=j is

$P(L= \varsigma \mid J=j, r ,k)$

$$= \{ (n-j)!/((n-j-\varsigma)!!!) \} \cdot \{ ((1-r)/(1-r \cdot C_{j+1}))^{n-j-\varsigma} \} \cdot \{r \cdot (1-C_{j+1})/(1-r \cdot C_{j+1})\}^{\varsigma} \quad (3.11.7)$$

which is derived from (3.11.7) by replacing m-j with ς. Hence, given $M \sim Bin(n,r)$, J=j and k, we have that L is binomially distributed with parameters n-j and $r \cdot (1-C_{j+1})/(1-r \cdot C_{j+1})$. So the expected number of defective items left after this burn-in procedure being stopped with j failed defective items observed is

$$E(L|J=j\ r,k) = (n-j) \cdot r \cdot (1-C_{j+1})/(1-rC_{j+1}). \quad (3.11.8)$$

It is clear that, given $M \sim Bin(n,r)$, the expected number of defectives left after burn-in is

$$E(L|r,k) = \sum_{j=0,m} P(J=j|r,k) \cdot E(L|j,r,k) \quad (3.11.9)$$

where

$$P(J=j|r,k) = \sum_{m=j,n} P(J=j, M=m|r k)$$

$$= \sum_{m=j,n} (n!/[(m-j)!\cdot(n-m)!)])\cdot(1-r)^{n-m}\cdot(1-C_j+1)^{m-j}\cdot r^{m}\cdot B_j.$$

CHAPTER IV

PROCEDURE II - LARGE SAMPLE THEORY

§4.1 Introduction

In this chapter, $\lim_{n\to\infty} P(R(t;D,m,n) \geq \rho)$ is investigated through $\lim_{m\to\infty} P(U_{m:1} < C_1(k), U_{m:2} < C_2(k), \cdots, U_{m:m_0} < C_{m_0}(k))$, where $U_{m:i}$ is the i-th uniform(0,1) order statistic from a sample of size m, $C_i(k)$ is defined as in chapter 3, and m is the assumed true number of defectives in a burn-in lot with size n and $0 < \lim_{m\to\infty} m/n = r < 1$. This investigation tells us the following results:

1. $\lim_{m\to\infty} P(U_{m:1} < C_1(k)) < 1.$ (4.1.1)

   Thus, $\lim_{m\to\infty} P(U_{m:1} < C_1(k), U_{m:2} < C_2(k), \cdots, U_{m:i} < C_i(k)) < 1$ for any i, $1 \leq i \leq m$.

   This tells us that $\lim_{n\to\infty} P(R(t;D,m,n) \geq \rho) \geq \alpha$ may not be achievable if we do not modify this stopping rule to prevent it from stopping at an early stage of burn-in. For $1 \leq i(m) \leq m^{\mu}$ with $\mu < 1/2$, we will see in §4.2 that the limit of the converted stopping time $\lim_{m\to\infty} \sqrt{m}\cdot(C_{i(m)}(k) - i(m)/m) = 0$. (Note: If $\limsup i(m)/\sqrt{m} = 0$, then $\lim_{m\to\infty} \sqrt{m}\cdot(C_{i(m)}(k) - i(m)/m)$ is still 0.) This limit tells us that the chance of stopping this screening procedure before the first $\sqrt{m}$ defective items have failed is high, i.e., the chance of stopping this rule before m_0 defective items are eliminated is high. In addition if $\lim_{m\to} i(m)/\sqrt{m}$ is a constant, then $\lim_{m\to\infty} \sqrt{m}\cdot(C_{i(m)}(k) - i(m)/m)$ is also a positive constant and

$\lim_{m\to\infty}P(U_{m:i(m)} < C_{i(m)}(k))=1$.

The limit, inequality (4.1.1), will be discussed in detail in Sections 4.2 and 4.3.

2. Let $j(m)$ and $\eta(m)$ be arbitrary functions of m satisfying

$$m^{\mu} < \eta(m) < m_0 - m^{\mu} < j(m) < m_0, \tag{4.1.2}$$

where μ is a constant with $1/2 < \mu < 1$.

If $k=\rho$,

$$\lim_{m\to\infty}P(U_{m:m^{\mu}}<C_{m^{\mu}}(k),U_{m:m^{\mu}+1}<C_{m^{\mu}+1}(k),\cdots,U_{m:\eta(m)}<C_{\eta(m)}(k))=1, \tag{4.1.3}$$

$$\lim_{m\to\infty}P(U_{m:j(m)}<C_{j}(k),U_{m:j(m)+1}<C_{j(m)+1}(k),\cdots,U_{m:m0}<C_{m0}(k)) < 1. \tag{4.1.4}$$

In addition, if $k > \rho$ then

$$\lim_{m\to\infty}P(U_{m:\eta(m)}<C_{\eta(m)}(k),U_{m:\eta(m)+1}<C_{\eta(m)+1}(k),\cdots,U_{m:m0}<C_{m0}(k))=1. \tag{4.1.5}$$

Equation (4.1.3) tells us that this stopping rule will not allow the burn-in process to stop before the number of defective items eliminated is very close to $m_0 - m^{\mu}$, if it has not stopped prior to stage m^{μ}. As in (1), the case $\mu=1/2$ is of interest to us and it will be studied in detail in this chapter. Inequality (4.1.4) will be used to find the value of the constant k so that $P(R(t;D,m,n)\geq\rho) \geq \alpha$ is ensured under this stopping rule. Based on (4.1.1), (4.1.3) and (4.1.4),when $k=\rho$, if we don't stop burn-in at an early stage then burn-in can only be stopped in a small neighborhood of m_0. The study of this neighborhood is the most important part of this chapter. In addition, Equation (4.1.5) tells us that if burn-in is not stopped at an early stage and k is more than ρ, then the limit of $P(R(t;D,m,n)\geq\rho)$ as m goes to infinity is 1.

Based on these results, two algorithms to find k are developed. Note from (4.1.4) and (4.1.5) that as $n \to \infty$, we must have $k=k(\rho,\alpha,m,n) \to \rho$. One of these two is quite easy to use to obtain k. This is the major application of the large sample theory of this procedure. Remember that the algorithm to find k in the previous chapter is very complicated. In addition to this, equation (4.1.1) tells us that our reliability goal

$P(R(t;D,m,n) \geq \rho) = P(U_{m:1} < C_1(k), U_{m:2} < C_2(k), \cdots, U_{m:m0} < C_{m0}(k)) \geq \alpha$

may not be achievable unless the early stopping bounds $C_i(k)$ are defined appropriately.

§ 4.2 Notation and Definitions

Let $U_1, U_2, \cdots, U_m$ denote independent uniform[0,1] random variables on [0,1], and define the uniform empirical distribution function G_m by

$$G_m(u) = (1/m)\cdot\Sigma_{i=1,m}1(U_i \le u) \quad \text{for } 0 \le u \le 1 \tag{4.2.1}$$

where

$$1(U_i \le u) = 1 \text{ if } U_i \le u, \text{ otherwise } 1(U_i \le u) = 0. \tag{4.2.2}$$

The inverse of the uniform empirical distribution function is defined as

$$G_m^{-1}(u) = \inf\{x: G_m(x) \ge u\} = U_{m:i} \quad \text{for } (i-1)/m \le u < i/m \text{ and } 1 \le i \le m \tag{4.2.3}$$

with $G_m^{-1}(0) = 0$, where

$$U_{m:1} \le U_{m:2} \le \cdots \le U_{m:m} \le U_{m:m+1} = 1 \tag{4.2.4}$$

denote the uniform order statistics. For $0 \le u \le 1$, define the uniform quantile process

$$V_m(u) = \sqrt{m}\cdot(G_m^{-1}(u) - I(u)) \tag{4.2.5}$$

and let

$V(u)$ denote a Brownian bridge. (4.2.6)

We have the following well-known results (The proof and the discussion of these results can be found in Billingsley (1968), or Csorgo (1983), or Shorack and Wellner (1986). The proofs will not be repeated here). For $0 \le u_1 \le u_2 \ldots \le u_i \le 1$,

$$\{V_m(u_1), V_m(u_2), \ldots, V_m(u_i)\} \to_d \{V(u_1), V(u_2), \ldots, V(u_i)\} \text{ as } m \to \infty, \tag{4.2.7}$$

i.e., V_m converges to V in finite dimensional distributions (Csorgo (1983, page 12)) Moreover, the quantile process, $V_m(u)$, converges to the Brownian bridge, $V(u)$, weakly in the space of discontinuous functions with right limits equipped with the Skorokhod topology. The Skorokhod representation implies that there exists a probability space with a sequence of Brownian bridges $\{V^{*m}(u); 0 \le u \le 1\}$ such that

$$\sup_{0 \le u \le 1} | V_m(u) - V^{*m}(u) | \longrightarrow_p 0 \tag{4.2.8}$$

as stated in Csorgo (1983, Equation (1.5.13)).

We now introduce some definitions which translate the index scale to the unit interval. Under the condition

$\text{limit}_{m\to\infty}(m/n)=r$,

for $\exp(-t) \le \pi \le 1$, define

$m0(\pi)= [m\cdot(1-\exp(-t))-n\cdot(1-\pi)]/(\pi-\exp(-t))$

$s_{m0} = s_{m0}(\pi) = m0(\pi)/m$ and

$s_0 = s_0(\pi) = \text{limit}_{m\to\infty}\, s_{m0}(\pi) = \text{limit}_{m\to\infty}(m0(\pi)/m)$.

We have s_0

$= \text{limit}_{m\to\infty}(1/m)\cdot\{[m\cdot(1-\exp(-t))-n\cdot(1-\pi)]/(\pi-\exp(-t))\}$

$= \{r\cdot(1-\exp(-t))-(1-\pi)\}/[r\cdot(\pi-\exp(-t))]$

$= \{(1-\exp(-t))-((1-\pi)/r)\}/(\pi-\exp(-t))$.

Note: $s_0(\pi)$ is an increasing function in π. (4.2.9)

If $\text{limit}_{m\to\infty}\, j(m)/m = s$, we have

$\text{limit}_{m\to\infty}\, C_{j(m)}(k)$

$= \text{limit}_{m\to\infty}\, j(m)\cdot(1-\exp(-t))/\{(n-j(m))\cdot(1-k)+j(m)\cdot(1-\exp(-t))\}$

$= \text{limit}_{m\to\infty}\, (j(m)/m)\cdot(1-\exp(-t))/\{(n/m)\cdot(1-k)+(j(m)/m)\cdot(k-\exp(-t))\}$

$= s\cdot(1-\exp(-t))/\{(1/r)\cdot(1-k)+s\cdot(k-\exp(-t)\}$. (4.2.10)

If $1 \le j(m) \le m^{\mu}$ with $0 \le \mu < 1$, we have

$\text{limit}_{m\to\infty}\, C_{j(m)}(k) = 0$.

This is a special case of (4.2.10), i.e., s=0.

Using the above results, we can prove that

$\text{limit}_{m\to\infty} P(R(t;D,m,n)\ge\rho)$

$= \lim_{m\to\infty} P(U_{m:1} < C_1(k), U_{m:2} < C_2(k), \cdots, U_{m:m0} < C_{m0}(k))$ (4.2.11)

is the boundary crossing probability of a Brownian bridge (quantile process). However, the mean and variance of V(0) are both 0, and the boundary at 0 is also 0. (See Equation (4.2.1) below.) So, we have the following difficulty: there is some unclear positive probability that the stopping rules, defined in Chapter III, stop before a sufficient proportion of defective items is removed. In order to avoid this difficulty, let's modify these rules by replacing the $C_i(k)$s with $C_{m^\mu}(k)$ if $1 \le i \le m^\mu$. So, we use the **Modified Stopping Rule**:

Stop burn-in at $C_i^*(k,\mu)$ when the first i is reached with $U_i > C_i^*(k,\mu)$, where

$$C_i^*(k,\mu) = \begin{cases} C_{m^\mu}(k) & \text{if } 1 \le i \le m^\mu, \\ C_i(k) & \text{if } m^\mu \le i \le n-1. \end{cases} \quad \text{(S.4.1)}$$

After this modification, the stopping times of Stopping Rule (S.4.1) will mainly depend on $m_0(k)/m$. We can see this property from Lemma 4.2.1 below. In Section 5.4, we will study the case that m is at least 320 and n is 4000 numerically.

Let $\text{limit}_{n\to\infty} m/n = r$. We have

$\text{limit}_{m\to\infty} \sqrt{m}\cdot(C_{j(m)}(k) - j(m)/m)$ (4.2.12)

$= \text{limit}_{n\to\infty} \sqrt{m}\cdot(j(m)/m)\cdot\{(1-\exp(-t))/\{((n-j(m))/m)\cdot(1-k)+(j(m)/m)\cdot(1-\exp(-t))\} - 1\}$

$= 0$, if $j(m) = o(\sqrt{m})$, (4.2.13)

$= \varsigma_1$ for some $-\infty < \varsigma_1 < \infty$ if $\text{limit}_{n\to\infty}(j(m)/\sqrt{m}) = \varsigma_1/\{r\cdot(1-\exp(-t))/(1-k) - 1\}$, (4.2.14)

$= \infty$ if $s_0(k) > \text{limit}_{n\to\infty} j(m)/m$ and $\lim\inf j(m)/\sqrt{m} = +\infty$, (4.2.15)

$= \varsigma_2$ for some $-\infty < \varsigma_2 < \infty$ if

$j(m) = m\cdot\{(1-\exp(-t) - (n/m)\cdot(1-k)\cdot(1+a/\sqrt{m})\}/[(k-\exp(-t))\cdot(1+a/\sqrt{m})]$

where $a = \varsigma_2/s_0(k)$ (see the following note), (4.2.16)

$= \quad -\infty, \quad \text{if } s_0(k) < \text{limit}_{n\to\infty}\, j(m)/m \le 1.$ (4.2.17)

Furthermore, it is clear that

$\varsigma_2 = 0$ if $\text{limit}_{n\to\infty}\ (j(m)/m - s_0(k)) = 0,$ (4.2.18)

$\varsigma_2 > 0$ if $\text{limit}_{n\to\infty}\ (j(m)/m - s_0(k)) < 0,$ (4.2.19)

$\varsigma_2 < 0$ if $\text{limit}_{n\to\infty}\ (j(m)/m - s_0(k)) > 0.$ (4.2.20)

Note: In order to ensure

$\text{limit}_{n\to\infty}\sqrt{m}\cdot(j(m)/m)\cdot\{(1-\exp(-t))/\{((n-j(m))/m)\cdot(1-k)+(j(m)/m)\cdot(1-\exp(-t)) - 1\} = \varsigma_2$

when $\text{limit}_{n\to\infty}\, j(m)/m = s_0(k)$, we need to have

$(1-\exp(-t))/\{(n/m)\cdot(1-k)+(j(m)/m)\cdot(k-\exp(-t)) - 1 = a/\sqrt{m}$ for $a = \varsigma_2/s_0(k)$.

That is

$j(m) = m\cdot\{(1-\exp(-t) - (n/m)\cdot(1-k)\cdot(1+a/\sqrt{m})\}/[(k-\exp(-t))\cdot(1+a/\sqrt{m})].$

In order to remove at least m_0 defective items, we don't want to stop before $s_0(\rho)$. To help achieve this goal, we replace all $C_j(k)$, $1 \le j \le \sqrt{m}$, by $C_m\mu(k)$, for $1/2 < \mu < 1$, since

$\text{limit}_{m\to\infty} \sqrt{m}\cdot(C_m\mu(k)-j(m)/m) = \infty$, when $\lim\sup j(m)/\sqrt{m} = 0.$ (4.2.21)

We can take $k > \rho$ to have $P(R(t; D, m, n) \ge \rho)$ converge to 1 as $m \to \infty$. That is, if $k > \rho$, the screening procedure defined by the modified stopping rule (S.4.1) will be forced to ignore the first $\sqrt{m}$ failure times and the probability that the ratio, J/m, reaches $s_0(k)$ will be approximately one (if m is large enough). Moreover, (4.2.9) implies that $s_0(k) > s_0(\rho)$ if $k > \rho$. Thus, we know that this screening procedure will stop after $s_0(\rho)$ with probability approximately one, i.e., the probability that $R(t; D, m, n) \ge \rho$ will be approximately one.

Using equation (4.2.21) and the facts that mean and variance of $V(0)$ are 0, we have

$\text{limit}_{m\to\infty} P(U_{m:1}<C_m^{\mu}(k), U_{m:2}<C_m^{\mu}(k), \cdots U_{m:m\mu}<C_m^{\mu}(k)) = 1$ for $0<\mu<1/2$.

Before proving Lemma 4.2.1, for $k \geq \rho$, let's define

$s_{0-}(k) = s_0(\rho) - (s_0(k) - s_0(\rho))/2$ and $s_{0+}(k) = s_0(k) + (s_0(k) - s_0(\rho))/2$.

It is clear that if $k > \rho$ then

$s_{0-}(k)<s_0(k)< s_{0+}(k)$ and $\lim_{k\to\rho} s_{0-}(k) = \lim_{k\to\rho} s_{0+}(k)$. (*)

The following lemma gives the boundary of Stopping Rule (S.4.1) in terms of $s_{0-}(k)$ and $s_{0+}(k)$.

Lemma 4.2.1:

For $k > \rho$,

$\liminf \sqrt{m}\bullet(C^*_{j(m)}(k)-j(m)/m) = +\infty$ if $0 < \lim (j(m)/m) \leq s_{0-}(k)$ and (4.2.22a)

$\limsup \sqrt{m}\bullet(C^*_{j(m)}(k)-j(m)/m) = -\infty$ if $1 > \lim (j(m)/m) \geq s_{0+}(k)$. (4.2.22b)

Proof.

If $0 < \text{limit}_{n\to\infty} j(m)/m \leq s_{0-}(k)$, then, by using (4.2.15), (*) and (4.2.19), we have $\liminf \sqrt{m}\bullet(C^*_{j(m)}(k)-j(m)/m) = +\infty$. The reason is that

$\liminf \sqrt{m}\bullet(C^*_{j(m)}(k)-j(m)/m) < +\infty$ may occur only when $\limsup j(m)/m = 0$ as seen from (4.2.13) and (4.2.14).

If $1 > \lim (j(m)/m) \geq s_{0+}(k)$, then, by using (4.2.17) and (*), we have $\limsup \sqrt{m}\bullet(C^*_{j(m)}(k)-j(m)/m) = -\infty$.

The proof of this lemma is completed.

Here, k (or $s_0(k)$) is the only parameter under our control. If k is too large, a lot of time will be wasted on extra burning-in. If k is too small, we might not be able to achieve our reliability goal. In the following sections, we shall try to use the previous lemma to find k(m) so that

$$P(\sqrt{m}\cdot(U_{m:j}-j/m) \le \sqrt{m}\cdot(C^*_j(k)-j/m),\ j=1,\cdots,m_0(\rho)) \ge \alpha \tag{4.2.25}$$

for sufficiently large m.

§4.3 Property of the Transformed Stopping Time as A Boundary For the Brownian Bridge

To investigate (4.2.25) in more detail, let's define

$d(s,k)=d(s,r,k)=s\bullet\{(1-\exp(-t))/[(1/r)\bullet(1-k)+s\bullet(k-\exp(-t))]-1\}$ for $0 \le s \le 1$. (4.3.1)

For $0\le s\le 1$, $m/n=r$ and $j = [m\bullet s]$, it's clear that

$C_j(k) - j/m$

$= j\bullet(1-\exp(-t))/\{n\bullet(1-k)+j\bullet(k-\exp(-t))\} - j/m$

$= (j/m)\bullet\{ (1-\exp(-t))/[(n/m)\bullet(1-k) + (j/m)\bullet(k-\exp(-t))] -1 \}$

$\sim s\bullet\{(1-\exp(-t))/[(1/r)\bullet(1-k)+s\bullet(k-\exp(-t))]-1\}$

$= d(s,k)$ for $0 \le s \le 1$.

The function $d(s,k)$ is a transformed stopping boundary of our unmodified stopping rule. For our modified stopping rule (S.4.1), define

$d^*(j/m,k) = \sqrt{m}\bullet(C^*_j(k) - j/m)$ for $1 \le j \le m$. We have

$\sqrt{m}\bullet d^*(s,k) \sim \sqrt{m}\bullet d(s,k)$ for $0 < s \le 1$.

Moreover, by using Lemma 4.2.1,

if $\text{limit}_{m\to\infty}\, i(m)/m = s < s_{0-}(k)$, then $\liminf (\sqrt{m})d(i(m)/m) = +\infty$.

If $\text{limit}_{m\to\infty}\, i(m)/m = s > s_{0+}(k)$, then $\limsup (\sqrt{m})d(i(m)/m) = -\infty$.

If $s_{0-}(k) < \text{limit}_{m\to\infty}\, i(m)/m = s < s_{0+}(k)$, then the limit behavior of $(\sqrt{m})d(i(m)/m)$ depends on how k is defined as a function of m.

In addition, the confidence of achieving our reliability goal, $R(t;D,m,n)\ge\rho$,

$P(R(t;D,m,n)\ge\rho)$

$= P(\sqrt{m}\bullet(U_{m:j}-j/m) < \sqrt{m}\bullet(C^*_j(k)-j/m),\ j=1,\cdots,m_0(\rho))$

is approximated by

$P(V(s) < \sqrt{(n \cdot r)} \cdot d(s,r,k), \text{ for } 0<s\leq s_0(\rho))$ (4.3.2)

In (4.3.2), we omit the point $s = 0$, because for $C^*_j(k)$, the limiting process cannot cross its boundary at $s=0$.

So, $\sqrt{m} \cdot d(s,k)$ is the boundary of interest to us.

Using Lemma 4.2.1 and Equation (4.3.2), we know that

$P(R(t;D,m,n)\geq\rho)$

$= P(V_m(j/m) < m^{1/2} \cdot d(j/m,m/n,k), \text{ for } 0<j/m\leq s_0(\rho))$

By using the weak convergence of V_m to V as described in (4.2.8) and the above discussion, we have

$P(R(t;D,m,n)\geq\rho)$

$\sim P(V(s) < \sqrt{(n \cdot r)} \cdot d(s,k), \text{ for } 0<s\leq s_0(\rho)) \sim 1$, if $k > \rho$, (4.3.3)

when Stopping Rule (4.2.1) is used, i.e., assuming

$P(V(0) < \lim_{m\to\infty \text{ and } j(m)/m\to 0} \sqrt{m} \cdot (C^*_{j(m)}(k)-j(m)/m)) = 1.$

In order to ensure $P(R(t;D,m,n)\geq\rho) = \alpha$, we can find an appropriate $k(\rho,\alpha,m,n)$ by solving

$P(V(s) < \sqrt{(n \cdot r)} \cdot d(s,r,k(\rho,\alpha,m,n)), \text{ for } 0<s\leq s_0(\rho)) = \alpha$, or

$P(V(s_0(\rho)) < \sqrt{(n \cdot r)} \cdot d(s_0(\rho),r,k(\rho,\alpha,m,n))) = \alpha.$ (4.3.4)

The reason is: $P(V(s) < \sqrt{(n \cdot r)} \cdot d(s,r,k(\rho,\alpha,m,n)), \text{ for } 0<s<s_0(\rho)) = 1$ for $k \geq \rho$. In addition, we will show in Section 4.4 how to choose $k(\rho,\alpha,m,n)$ so that $P(R(t;D,m,n)\geq\rho) \geq \alpha$.

Before giving the procedure to find an appropriate k, let's investigate some properties of $d(s,r,k)$.

Lemma 4.3.1:

(1) For $0 \le r, k \le 1$, $d(s,r,k) > 0$ if and only if $0 < s < s_0(k)$.

(2) $d(s,r,k)$ is an increasing function in k and r.

Note: We are interested in the case $d(s,r,k) > 0$, i.e., $\text{limit}_{m\to\infty} \sqrt{(n\cdot r)}\cdot d(s,r,k) = \infty$, because we'd like to have our reliability $\text{limit}_{m\to\infty} P(R(t;D,m,n) \ge \rho)$ close to 1 (exceeding or equaling α).

Proof:

Since $s > 0$, to prove $d(s,r,k) > 0$, we only need to show

$$(1-\exp(-t))/\{(1/r)\cdot(1-k)+s\cdot(k-\exp(-t))\}-1 > 0. \tag{4.3.5}$$

Inequality (4.3.5) is true

$\iff$ $1-\exp(-t)) > (1/r)\cdot(1-k)+s\cdot(k-\exp(-t))$

$\iff$ $s < \{(1-\exp(-t)) - (1/r)\cdot(1-k)\}/(k-\exp(-t))$.

It is clear that $d(s,r,k)$ is an increasing function in r and k. The proof is completed.

Note:

1. For $0 < s < 1$, we are only interested in the case that

$$0 < \{(1-\exp(-t)) - (1/r)\cdot(1-k)\}/(k-\exp(-t)) < 1. \tag{4.3.6}$$

The right hand side of (4.3.6) is true

$\iff \{(1-\exp(-t)) - (1/r)\cdot(1-k)\} < (k-\exp(-t))$

$\iff 1-\exp(-t) - 1/r + (1/r)\cdot k < k -\exp(-t)$

$\iff k\cdot(1/r -1) < 1/r - 1$

$\iff k < 1$.

$k < 1$ is always true, otherwise, at $k=1$, $C_j(1) = 1$ for all j, i.e. never stop burn-in until all items are removed from burn-in lot.

The left hand side of (4.3.6) is true

$\iff 1 - \exp(-t) > (1/r)\cdot(1-k)$

$\iff r > (1 - k)/(1 - \exp(-t))$.

This tells us that no burn-in is required if $r < (1 - k)/(1 - \exp(-t))$.

2. For any fixed r, if $k1 < k2$, then $d(s,r,k1) < d(s,r,k2)$. This tells us that for two burn-in lots from the same production line. The lot using larger k will have longer burn-in duration and higher $P(R(t;D,m,n) \geq \rho)$ if n is large enough.
3. For any fixed k and r, $d(s1,r,k) < d(s2,r,k)$ if $s1 < s2$. This tells us that, when the same stopping rule, same k, is used for the production lots from the same production line, more time is required to screen out more defective items.
4. For any fixed k, $d(s,r1,k) < d(s,r2,k)$ if $r1 < r2$. Therefore, if the same k is used for different burn-in lots from different production lines, more burn-in time is required for the lot with a larger proportion of defective items in it. In this case, more defective items must be eliminated through burn-in for the lot with more defectives.

In order to have $\lim_{n\to\infty} P(R(t;D,m,n)\geq\rho) \geq \alpha$, we must have $s_0(k) \geq s_0(\rho)$. The following theorem tells us when this is true in terms of k.

Lemma 4.3.2:

When burn-in is required,

$d(s_0(\rho),r,k) > 0$ (or < 0) if and only if $k > \rho$ or ($< \rho$). (4.3.7)

Moreover, denote $s_{m0} = s_{m0}(\rho)$, and take $\varepsilon > 0$.

If $k - \rho \geq \varepsilon$, then $\text{limit}_{m\to\infty} \sqrt{m}\cdot d(s_{m0},r,k) = \infty$. (4.3.8a)

If $k - \rho \leq -\varepsilon$, then $\text{limit}_{m\to\infty} \sqrt{m}\cdot d(s_{m0},r,k) = -\infty$. (4.3.8b)

(Here, $\text{limit}_{m\to\infty}\ s_{m0} = s_0$.)

In addition to this. for $k \geq \rho$ and $1/2 < \mu < 1$, $m\rho - m^{\mu} > j1(m) > m^{\mu}$ and $\liminf j2(m)/\sqrt{m} = 0$, we have

$$\text{limit}_{m\to\infty}\sqrt{m}\cdot d(j1(m)/m,r,k) = \infty \tag{4.3.9a}$$

$$\text{limit}_{m\to\infty}\sqrt{m}\cdot d(j2(m)/m,r,k) = 0. \tag{4.3.9b}$$

Note:

1. Equation (4.3.8a) says that our reliability will be achieved with probability 1 if $k > \rho$.

   Equation (4.3.8b) says that with probability 1 our reliability will not be achieved if $k < \rho$.

2. Equation (4.3.9a) tells us that if $k \geq \rho$, with probability approaching one, as m incresaes, $T_j < S(j,k)$, where $m0\text{-}m^{\mu} > j(m) > m^{\mu}$ for $1/2 < \mu < 1$. Equation (4.3.9b) indicates the range of j(m) where we are not very sure about the probability performance of this stopping rule.

Proof:

We have $s_{m0}(\rho) = s_{m0} > 0$ when burn-in is required. In addition,

$d(s_{m0}) = s_{m0}\cdot\{(1-\exp(-t))/[(1/r)\cdot(1-k) + s_{m0}\cdot(k-\exp(-t))]-1\}$ and

$d(s_0) = \text{limit}_{n\to\infty}\, d(s_{m0}) = s_0\cdot\{(1-\exp(-t))/[(1/r)\cdot(1-k) + s_0\cdot(k-\exp(-t))]-1\}$

where $s_0 = s_0(\rho)$.

So, we only need to show that

$$\{1-\exp(-t)) - (1/r)\cdot(1-k)\}/(k-\exp(-t)) > s_0 \iff k > \rho. \tag{4.3.10}$$

The left hand side of (4.3.10) is true

$\iff \{(1-\exp(-t)) - (1/r)\cdot(1-k)\}/(k-\exp(-t)) > \{(1-\exp(-t)) - (1/r)\cdot(1-\rho)\}/(\rho-\exp(-t))$

$\iff (\rho-\exp(-t))\cdot\{(1-\exp(-t)) - (1/r)\cdot(1-k)\} > (k-\exp(-t))\cdot\{1-\exp(-t)) - (1/r)\cdot(1-\rho)\}$

$\iff ((1-\exp(-t))-(1-\rho))\cdot\{(1-\exp(-t))-(1/r)\cdot(1-k)\} > ((1-\exp(-t))-(1-k))\cdot\{(1-\exp(-t))-(1/r)\cdot(1-\rho)\}$

$\iff -(1-\exp(-t))\cdot(1/r)\cdot(1-k)-(1-\rho)\cdot(1-\exp(-t))$
$\quad > -(1-\exp(-t))\cdot(1/r)\cdot(1-\rho)-(1-k)\cdot(1-\exp(-t))$

$\Longleftrightarrow (1-\exp(-t))\cdot(1/r)\cdot(k-\rho) > (1-\exp(-t))\cdot(k-\rho)$

$\Longleftrightarrow (1/r)\cdot(k-\rho) > (k-\rho)$

$\Longleftrightarrow (k-\rho) > 0$, since $1/r > 1$.

Equations (4.3.8), (4.3.9a), (4.3.9b), (4.3.9c) are trivial results of the above derivation. The proof of this lemma is completed.

Note:

1. For $1 \le i(m) = o(\sqrt{m})$ and $j(m) \le i(m)$, (4.3.9c) tells us that $\sqrt{m}\cdot(C_{m0}(k)-j(m)/m)$ converges to 0 as $m \to \infty$. In addition, the mean and variance of $\sqrt{m}\cdot(U_{m:j}-j/m)$ go to 0. So, the probability,

   $\text{limit}_{m\to\infty} P(\sqrt{m}\cdot(U_{m:j}-j/m) < \sqrt{m}\cdot(C_j(k)-j/m), j=1,2,\cdots,i(m))$

   is not clearly given by the large sample theory, as mentioned before. To have a better picture of

   $\text{limit}_{m\to\infty} P(\sqrt{m}\cdot(U_{m:j}-j/m) < \sqrt{m}\cdot(C_j(k)-j/m), j=1,2,\cdots,i(m))$,

   we'll study this limit for small "i(m)"s in section 4.7. How can we avoid this difficulty? Remember that

   $R(t;D,m,n) \ge \rho$

   is true only when at least $m_0(\rho)$ failed defectives are observed and the $m_0(\rho)$ is of order m. Therefore, we can avoid this difficulty by making the above probability 1 by increasing the boundary $C_i(k)$'s. Using (4.3.9b), $C_i(k)$ can be modified as in our modified Stopping Rule (S.4.1).

2. When $k = \rho$, $\text{limit}_{m\to\infty}\sqrt{m}\cdot d(j(m)/m)$ goes from positive to negative as $j(m)$ crosses $m_0(k)$. Clearly, for $1/2 < \mu < 1$, $\text{limit}_{m\to\infty}\sqrt{m}\cdot d(j(m)/m) = \infty$ if $\liminf j(m)/\sqrt{m} = 0$ and $j(m) \le m_0 - m^{\mu}$. In addition, we have

   $\text{limit}_{m\to\infty}\sqrt{m}\cdot d(j(m)/m) = -\infty$

if $n-1 \geq m > j(m)$, and $\lim\inf m^{-\mu}\{j(m)-m_0\} = +\infty$.

§ 4.4 Approximating k via Large Sample Theory

An equation to obtain an appropriate "k" is obtained in this section. This equation is based on the large sample theory developed in the previous sections, especially Lemma 4.2.1 and Equation (4.3.4), when a suitable boundary for this screening process at $s_{m0}(k)$ is given. We know that, for Stopping Rule (S.4.1)

$\lim_{n\to\infty} P(R(t;D,m,n)\geq\rho)=1$ for $k > \rho$ and

$\lim_{n\to\infty} P(R(t;D,m,n)\geq\rho)=0$ for $k < \rho$.

In addition, equation (4.3.4) tells us that

$\lim_{n\to\infty} P(R(t;D,m,n)\geq\rho) = \alpha$

can be achieved if we can find an appropriate $k(\rho,\alpha,m,n)$ such that

$P(V(s_0(\rho)) < \sqrt{(n\cdot r)}\cdot d(s_0(\rho),r,k(\rho,\alpha,m,n))),\ 0 < s \leq s_0) = \alpha$.

So, we only need to obtain a $k(\rho,\alpha,m,n)$ with $P(\ V(s) < h^-(s),\ 0 < s \leq s_0) = \alpha$.

We have, for any fixed s,

$$V(s) \sim N(0,\{\sqrt{(s\cdot(1-s))}\}^2) \qquad \text{or} \qquad V(s)/\sqrt{(s\cdot(1-s))} \sim N(0,1). \tag{4.4.1}$$

For any α in (0,1), we can easily find a number $b=b(\alpha)$ in $(-\infty,\infty)$ such that, for any fixed s with $0<s<1$,

$$P(\ V(s)/\sqrt{(s\cdot(1-s))} \leq b(\alpha)) = \alpha. \tag{4.4.2}$$

In terms of V(s) and $\sqrt{(n\cdot r)}\cdot d(s,k)$, at $s_0(\rho)$, we'd like to have, for $n \to \infty$ and $m/n \to r$,

$$\lim_{n\to\infty} \{\sqrt{(n\cdot r)}\}\cdot d(s_{m0}(\rho),r,k(\rho,\alpha,m,n)) /\sqrt{(s_{m0}(\rho)\cdot(1-s_{m0}(\rho)))} = b(\alpha). \tag{4.4.3}$$

Based on these three relations (4.4.1), (4.4.2) and (4.4.3), Lemma 4.4.1 tells us how to find k. In addition, Theorem 4.4.1 tells us that our reliability goal will be ensured if the value of k obtained by Lemma 4.4.1 is used.

Lemma 4.4.1:

When m is large enough, for any real number b, we will have

$$\{\sqrt{(n\cdot r)}\}\cdot d(s_{m0}(\rho),r,k(\rho,\alpha,m,n)) \sim b\cdot\sqrt{(s_{m0}(\rho)\cdot(1-s_{m0}(\rho)))} \quad (4.4.4)$$

if and only if

$$k(\rho,\alpha,m,n) \sim \rho + r\cdot\{(\rho - \exp(-t))/(1 - r)\}\cdot[b/\{\sqrt{[m\cdot s_0/(1-s_0)]}+b\}] \quad (4.4.5)$$

where $r=m/n$ and $s_0 = s_0(\rho) = \text{limit}_{m->\infty}s_{m0}(\rho)$.

Proof:

(Lemma 4.3.2 and its following notes will be used in this proof.)

Let $s_{m0} = s_{m0}(\rho)$ and $d(s_{m0}) = d(s_{m0}(\rho),r,k(\rho,\alpha,m,n))$. We have

$\{\sqrt{(n\cdot r)}\}\cdot d(s_{m0}(\rho),r,k(\rho,\alpha,m,n)) \sim b\cdot\sqrt{(s_{m0}(\rho)\cdot(1-s_{m0}(\rho)))}$

$\Leftrightarrow d(s_{m0}) \sim \{\sqrt{(s_{m0}\cdot(1-s_{m0}))}\}\cdot b/\sqrt{m}$.

In addition

$\text{limit}_{m->\infty}d(s_{m0}) = s_0\cdot\{(1-\exp(-t))/[(1/r)\cdot(1-k)+s_0\cdot(k-\exp(-t))]-1\}$.

Using these two equations, for m being large enough, we have

$d(s_0) \sim \{\sqrt{((1-s_0)\cdot s_0)}\}\cdot b/\sqrt{m}$

$(1-\exp(-t))/\{(1/r)\cdot(1-k) + s_0\cdot(k-\exp(-t))\}-1 \sim \{\sqrt{((1-s_0)/s_0)}\}\cdot b/\sqrt{m}$

$(1-\exp(-t))-(1/r)\cdot(1-k)-s_0\cdot(k-\exp(-t))\sim\{(1/r)\cdot(1-k)+s_0\cdot(k-\exp(-t))\}\cdot\{\sqrt{((1-s_0)/s_0)}\}\cdot b/\sqrt{m}$

$k\cdot(1/r - s)+1-\exp(-t)-1/r + s_0\cdot\exp(-t)$

$\sim -k\cdot(1/r - s)\cdot\{\sqrt{((1-s_0)/s_0)}\}\cdot b/\sqrt{m} + (1/r -s\cdot\exp(-t))\cdot \{\sqrt{((1-s_0)/s_0)}\}\cdot b/\sqrt{m}$

$k\cdot(1/r-s_0)\cdot\{1+[\sqrt{((1-s_0)/s_0)}]\cdot b/\sqrt{m}\}$

$\sim (1/r-s_0\cdot\exp(-t))\cdot(1+[\sqrt{((1-s_0)/s_0)}]\cdot b/\sqrt{m}\}-(1-\exp(-t))$

$$k \sim (1/r -s_0\cdot\exp(-t))/(1/r -s_0) - (1-\exp(-t))/\{(1/r -s_0)\cdot(1+[\sqrt{((1-s_0)/s_0)}]\cdot b/\sqrt{m})\}. \quad (4.4.6)$$

Further more, we have

$$(1/r -s_0\cdot\exp(-t))/(1/r -s_0) = (1 - r\cdot s_0\cdot\exp(-t))/(1 - r\cdot s_0), \quad (4.4.7)$$

$1 - r\cdot s_0\cdot\exp(-t)$

$= 1 - r\cdot\{r\cdot(1-\exp(-t))-(1-\rho)\}\cdot\exp(-t)/\{r\cdot(\rho-\exp(-t))\}$

$= \{(\rho-\exp(-t)) - r\cdot(1-\exp(-t))\cdot\exp(-t) + (1-\rho)\cdot\exp(-t)\}/(\rho-\exp(-t))$

$= \{\rho\cdot(1-\exp(-t)) - r\cdot(1-\exp(-t))\cdot\exp(-t)\}/(\rho-\exp(-t))$

$= (1 - \exp(-t))\cdot(\rho-r\cdot\exp(-t))/(\rho-\exp(-t))$, and

$1 - r\cdot s_0$

$= 1 - r\cdot\{r\cdot(1-\exp(-t))-(1-\rho)\}/\{r\cdot(\rho-\exp(-t))\}$

$= \{\rho-\exp(-t) - r\cdot(1-\exp(-t)) + (1-\rho)\}/(\rho-\exp(-t))$

$= (1 - \exp(-t))\cdot(1- r)/(\rho-\exp(-t))$.

So,

$(1 - r\cdot s_0\cdot\exp(-t))/(1 - r\cdot s_0)$

$= \{(1 - \exp(-t))\cdot(\rho-r\cdot\exp(-t))/(\rho-\exp(-t))\}/\{(1 - \exp(-t))\cdot(1- r)/(\rho-\exp(-t))\}$

$= (\rho-r\cdot\exp(-t))/(1- r)$

$= \rho + r\cdot(\rho - \exp(-t))/(1 - r)$. (4.4.8)

In addition,

$(1-\exp(-t))/[(1/r - s_0)\cdot(1+\{\sqrt{((1-s_0)/s_0)}\}\cdot b/\sqrt{m})]$

$= [r\cdot\{(\rho-\exp(-t))/(1-r)\}]/(1+\{\sqrt{((1-s_0)/s_0)}\}\cdot b/\sqrt{m})$ (4.4.9)

So, using (4.4.6) ~ (4.4.9), we have

$k(\rho,\alpha,m,n) = k$

$\sim \rho + [r\cdot(\rho - \exp(-t))/(1 - r)]\cdot[1 - 1/(1+\{\sqrt{((1-s_0)/s_0)}\}\cdot b/\sqrt{m})]$

$= \rho + [r\cdot(\rho-\exp(-t))/(1- r)]\cdot[\{\sqrt{((1-s_0)/s_0)}\}\cdot b/\sqrt{m}/(1+\{\sqrt{((1-s_0)/s_0)}\}\cdot b/\sqrt{m})]$

$= \rho + [r\cdot(\rho - \exp(-t))/(1 - r)]\cdot[b/\{\sqrt{[m\cdot s_0/(1-s_0)]}+b\}]$ (4.4.10)

The proof of this lemma is completed.

Note:

In order to study $k(\rho,\alpha,m,n)$, we'd like to write it as a function of $r=m/n$, i.e., $k(r)$. (The behavior of $k(r)$ will be studied in the following section.) Denote $s_0 = s_0(\rho) = \{r\cdot(1-\exp(-t)) - (1-\rho)\}/\{r\cdot(\rho-\exp(-t))\}$. We have

$s_0/(1-s_0) = \{r\cdot(1-\exp(-t)) - (1-\rho)\}/\{(1-\rho)\cdot(1-r)\}$.

In addition, if we define

$$k(r) = \rho + (\rho-\exp(-t))\cdot b/\{b\cdot((1/r)-1)+\sqrt{[(n/(1-\rho))\cdot[(1-\exp(-t)) - ((1-\rho)/r)} \\ \overline{- (r\cdot(1-\exp(-t)) + (1-\rho)]]}\},$$

we can see that $k(r) \sim k(\rho,\alpha,m,n)$.

From (4.4.10),

$$k \sim \rho + [r\cdot(\rho - \exp(-t))/(1 - r)]\cdot\{b/\{\sqrt{[n\cdot r\cdot\{r\cdot(1-\exp(-t)) - (1-\rho)\}/} \\ \overline{[(1-\rho)\cdot(1-r)]]}+b\}\}$$

$$= \rho + (\rho - \exp(-t))\cdot b/\{b\cdot(1-r)/r + \sqrt{[n\cdot[(1-r)/r]\cdot[r\cdot(1-\exp(-t)) - (1-\rho)]/(1-\rho)]}\}.$$

$$= \rho + (\rho-\exp(-t))\cdot b / \{b\cdot((1/r)-1) + \sqrt{[(n/(1-\rho))\cdot[(1-\exp(-t)) - ((1-\rho)/r) -} \\ \overline{r\cdot(1-\exp(-t)) +(1-\rho)]]}\}$$

$$= k(r). \tag{4.4.11}$$

2. The square root term in $k(r)$ is positive if $r > (1-\rho)/(1-\exp(-t))$. Its denominator is positive if $(1-\rho)/(1-\exp(-t)) < r < 1$ which is the range of r of interest to us (burn-in is required if $(1-\rho)/(1-\exp(-t)) < r < 1$).
3. $k(r) \geq \rho$ when n is sufficiently large and burn-in is required. This is a trivial result from (4.4.11).

From Lemma 4.4.1, we have the following theorem which tells us that our reliability goal, $P(R(t;D,m,n)\geq\rho)\geq\alpha$, is achieved if the value of k is obtained by using Equation (4.4.15). Before proving this theorem, let's define

Z: a standard normal random variable, (4.4.12)

z_α is the upper $100 \cdot \alpha$ percentile of Z, (4.4.13)

$$b(\alpha,m) = z_\alpha / \sqrt{(s_{m0}(\rho) \cdot (1-s_{m0}(\rho))}, \tag{4.4.14}$$

$$k(\rho,\alpha,m,n) \sim \rho + r \cdot \{(\rho - \exp(-t))/(1 - r)\} \cdot$$
$$[b(\alpha,m)/\{\sqrt{[m \cdot s_0/(1-s_0)]} + b(\alpha,m)\}]. \tag{4.4.15}$$

Theorem 4.4.1:

Under Stopping Rule (S.4.1), if k is defined by Equation (4.4.15), then

$$\mathrm{limit}_{n->\infty} P(\, R(t;D,m,n) \geq \rho\,) = \alpha \tag{4.4.16}$$

Proof:

Under the Stopping Rule (S.4.1), we will ignore the first m^μ, $1/2 < \mu < 1$, $U_{j:m}$'s (the first m^μ failure times of the defective items), that is

$\mathrm{limit}_{n->\infty} P(R(t;D,m,n) \geq \rho) = P(V(s) \leq \sqrt{(n \cdot r)} \cdot d(s,k),\ 0<s<s_0(\rho))$

$= P(V(s_0(\rho)) \leq \sqrt{(n \cdot r)} \cdot d(s_0(\rho)))$

$= \mathrm{limit}_{n->\infty} P(V(s_{m0}(\rho)) \leq \sqrt{(n \cdot r)\}} \cdot d(s_{m0}(\rho),r,k(\rho,\alpha,m,n)))$

Using

$\sqrt{(n \cdot r)\}} \cdot d(s_{m0}(\rho),r,k(\rho,\alpha,m,n)) \sim b(\alpha,m) \cdot \sqrt{(s_{m0}(\rho) \cdot (1-s_{m0}(\rho))},$

the expression above is equal to

$\mathrm{limit}_{n->\infty} P(V(s_{m0}(\rho)) \leq b(\alpha,m) \cdot \sqrt{(s_{m0}(\rho) \cdot (1-s_{m0}(\rho))}\,) =$ [illegible] $\leq z_\alpha\,) = \alpha.$

The proof of this theorem is completed.

§4.5 The Behavior of k(r)

The following Lemma tells us the behavior of k(r). From the discussion below, we can see that k(r) is a U-shaped function for $0 < (1-\rho)/(1-\exp(-t)) < r < 1$. Note: The behavior of k(r) for r from 0 to $(1-\rho)/(1-\exp(-t))$ is not given, because no burn-in is required if r is in this range.

Lemma 4.5.1:

Define: $r0 = (1-\rho)/(1-\exp(-t))$.

There exists a unique value r_n such that

k(r) decreases on $[r0, r_n]$ and increases on $[r_n, 1]$.

Moreover, $\lim_{n\to\infty} r_n = \{(1-\rho)/(1-\exp(-t))\}^{1/2} = \sqrt{r0}$.

Proof:

Define

$$g(r) = b\bullet((1/r)-1) + \{\sqrt{(n/(1-\rho))}\}\bullet\sqrt{[(1-\exp(-t))-((1-\rho)/r)-r\bullet(1-\exp(-t))+(1-\rho)]}. \quad (4.5.1)$$

Function g(r) is the only part of k(r) which has r in it and g(r) is the denominator in the second summation term of k(r). The function k(r) increases as g(r) decreases, and vice versa. Let's investigate g'(r) first.

$$g'(r)=-b/r^2+(1/2)\bullet\sqrt{(n/(1-\rho))}\bullet[(1-\rho)/r^2-(1-\exp(-t))]/\sqrt{[(1-\exp(-t))-(1-\rho)/r-r\bullet(1-\exp(-t))+(1-\rho)]}. \quad (4.5.2)$$

The sign of g'(r) is mainly determined by the second summation if n is large enough. For $0 < r < 1$, g'(r) is negative if its second summation is negative, since its first summation is negative. The second term of g'(r) is negative if and only if

$$(1-\rho)/r^2 < (1-\exp(-t)) \quad (4.5.3)$$

$$(1-\exp(-t))-(1-\rho)/r-r\bullet(1-\exp(-t))+(1-\rho) > 0. \quad (4.5.4)$$

Inequality (4.5.3) is true if and only if

$r > \sqrt{\{(1-\rho)/(1-\exp(-t))\}} = \sqrt{r_0}$. (4.5.6)

Inequality (4.5.4) is true if and only if

$(1-r)\cdot\{r\cdot(1-\exp(-t)) - (1-\rho)\}/r > 0 \iff$

$r > r_0$. (4.5.7)

Recall that (4.5.7) is the necessary condition for burn-in.

So, for $\sqrt{r_0} < r < 1$, k(r) increases (or g(r) decreases) as r increases.

For $0 \le r < r_0$, burn-in is not required.

For $r_0 < r < \sqrt{r_0}$, it is clear that k(r) decreases (or g(r) increases) as r increases, but the behavior of k(r) is not clear for r near $\sqrt{r_0}$.

Let's investigate the behavior of k(r) in a neighborhood of $\sqrt{r_0}$.

The derivative g'(r) is positive

$\iff (1/2)\cdot\sqrt{(n/(1-\rho))}\cdot[(1-\rho)/r^2-(1-\exp(-t))]/\sqrt{[(1-\exp(-t))-(1-\rho)/r-r\cdot(1-\exp(-t))+(1-\rho)]}$

$\ge b/r^2$

$\iff (1-\rho)-(1-\exp(-t))\cdot r^2 \ge 2\cdot b\cdot\sqrt{((1-\rho)/n)}\cdot\sqrt{[(1-\exp(-t))-(1-\rho)/r-r\cdot(1-\exp(-t))+(1-\rho)]}$

Define

$w(r) = (1-\exp(-t))-(1-\rho)/r-r\cdot(1-\exp(-t))+(1-\rho)$. (4.5.8)

We have

$g'(r) \ge 0$

$\iff (1-\rho)-(1-\exp(-t))\cdot r^2 \ge 2\cdot b\cdot\sqrt{((1-\rho)/n)}\cdot\sqrt{w(r)}$, (4.5.9)

$w((1-\rho)/(1-\exp(-t))) = 0$,

$w(\sqrt{(1-\rho)/(1-\exp(-t))}) = (1 - \exp(-t)) - \sqrt{(1-\exp(-t))}\cdot\sqrt{(1-\rho)} - \sqrt{(1-\rho)}\cdot\sqrt{(1-\exp(-t))}+(1-\rho)$

$= (\sqrt{(1-\exp(-t))} - \sqrt{(1-\rho)})^2 > 0$,

The derivative $w'(r) = (1-\rho)/r^2 - (1-\exp(-t))$ is positive and decreasing for

$(1-\rho)/(1-\exp(-t)) < r < \{(1-\rho)/(1-\exp(-t))\}^{1/2}$. So, $\sqrt{w(r)}$ is positive and increasing for

$(1-\rho)/(1-\exp(-t)) < r < \sqrt{(1-\rho)/(1-\exp(-t))}$. In addition, $(1-\rho)-(1-\exp(-t))\cdot r^2$, decreases as r increases from $(1-\rho)/(1-\exp(-t))$ to $\{(1-\rho)/(1-\exp(-t))\}^{1/2}$. So there is a unique $r_n > 0$ such that $g'(r_n) = 0$. Here, r_n is the point where g(r) achieves its maximum and where k(r) has its (local) minimum. In addition, as $n \to \infty$, the inequality (4.5.8a) can be written as $(1-\rho)-(1-\exp(-t))\cdot r^2 \geq 0$. So, $r_n \to \{(1-\rho)/(1-\exp(-t))\}^{1/2}$, as $n \to \infty$. The proof of this lemma is completed.

The definition of k(r) implies that

$$\lim_{r \to 1} k(r) \tag{4.5.10}$$

$$= \rho + \lim_{r \to 1} (\rho - \exp(-t))\cdot b/\{b\cdot(1-r)/r + \sqrt{[n\cdot[(1-r)/r]\cdot[r\cdot(1-\exp(-t)) - (1-\rho)]/(1-\rho)]}\}$$

$$= +\infty.$$

In addition,

$$k(r0) = 1. \tag{4.5.10a}$$

The range of r such that $\rho < k(r) < 1$ is the range of our interest. If $k(r) \geq 1$, then the screening procedure designed in this and the previous chapter will never be stopped. (In order to achieve our reliability goal, k(r) was required to be greater than ρ.)

Lemma 4.5.2:

Assume r0 < 1/2. For a fixed and sufficiently large n, the appropriate range of r for us to use k(r) as our k in this burn-in procedure is from r0 to min{r2,1} where r2 is the largest solution of k(r)=1.

Proof:

First, for a fixed n, let's solve $k(r) = 1$.

$$1 - \rho = (\rho-\exp(-t))\cdot b/\{b\cdot(1-r)/r + \sqrt{[(n/(1-\rho))\cdot[(1-r)/r]\cdot[r\cdot(1-\exp(-t))-(1-\rho)]]}\}$$

$$(1-\rho)\cdot\{b\cdot(1-r) + \sqrt{[(n\cdot(1-\rho))\cdot[(1-r)\cdot r]\cdot[r\cdot(1-\exp(-t))-(1-\rho)]]}\} = (\rho-\exp(-t))\cdot b\cdot r$$

$b \cdot [r \cdot (1-\exp(-t)) - (1-\rho)] = \sqrt{[(n \cdot (1-\rho)) \cdot [(1-r) \cdot r] \cdot [r \cdot (1-\exp(-t))-(1-\rho)]]}$

$b^2 \cdot [r \cdot (1-\exp(-t)) - (1-\rho)]^2 = (n \cdot (1-\rho)) \cdot [(1-r) \cdot r] \cdot [r \cdot (1-\exp(-t))-(1-\rho)]$ (4.5.11)

$b^2 \cdot [r \cdot (1-\exp(-t)) - (1-\rho)] = n \cdot (1-\rho) \cdot [(1-r) \cdot r]$

$b^2 \cdot [r \cdot (1-\exp(-t))/(1-\rho) - 1] = n \cdot (r-r^2)$

$r^2 - r \cdot [1- b^2 \cdot (1-\exp(-t))/((1-\rho) \cdot n)] - b^2/n = 0.$ (4.5.12)

Define

$B = 1- b^2 \cdot (1-\exp(-t))/((1-\rho) \cdot n)$, $C = - b^2/n$,

$r1 = [B - \sqrt{(B^2 - 4 \cdot C)}]/2$ and $r2 = [B + \sqrt{(B^2 - 4 \cdot C)}]/2$.

r1 and r2 are the two roots of equation (4.5.12).

It's trivial that $r1 < 0 < (1-\rho)/(1-\exp(-t))$. (4.5.13a)

$r2 = \{1- b^2 \cdot (1-\exp(-t))/((1-\rho) \cdot n) + \sqrt{([1- b^2 \cdot (1-\exp(-t))/((1-\rho) \cdot n)]^2 + 4 \cdot b^2/n)}\}/2$

$= \{1- b^2 \cdot (1-\exp(-t))/((1-\rho) \cdot n) + \sqrt{\{[1- (b^2/n)^2 \cdot [(1-\exp(-t))/(1-\rho))-2]]^2 + 4 \cdot (b^2/n)^2 \cdot}$

$((1-\exp(-t))/((1-\rho)-1)\}\}/2$

$< \{1- b^2 \cdot (1-\exp(-t))/((1-\rho) \cdot n) + \{[1- (b^2/n) \cdot ((1-\exp(-t))/(1-\rho))-2] + 2 \cdot (b^2/n) \cdot$

$\sqrt{((1-\exp(-t))/((1-\rho)-1)}\}\}/2$

$= 1- b^2 \cdot (1-\exp(-t))/((1-\rho) \cdot n) + (b^2/n) + (b^2/n) \cdot \sqrt{((1-\exp(-t))/((1-\rho)-1)}$

$= 1- (b^2/n) \cdot \{[(1-\exp(-t))/((1-\rho) - 1] - \sqrt{[(1-\exp(-t))/((1-\rho)-1]}\}$ (4.5.13b)

< 1 if $(1-\exp(-t))/((1-\rho) > 2$ when burn-in is required.

The above results implicitly tell us that (4.5.11) has exactly three roots: one is r1, one is r0 and the last one is r2. For n large enough, we have

$r1 < 0 < r0 < r2 < 1.$ (4.5.14)

So, this lemma is proved.

The relation between k(r) and r is sketched in Figure 4 for $r0 \leq r \leq r2$. The function k(r) has its minimum at r_n and k(r0)=k(r2)=1.

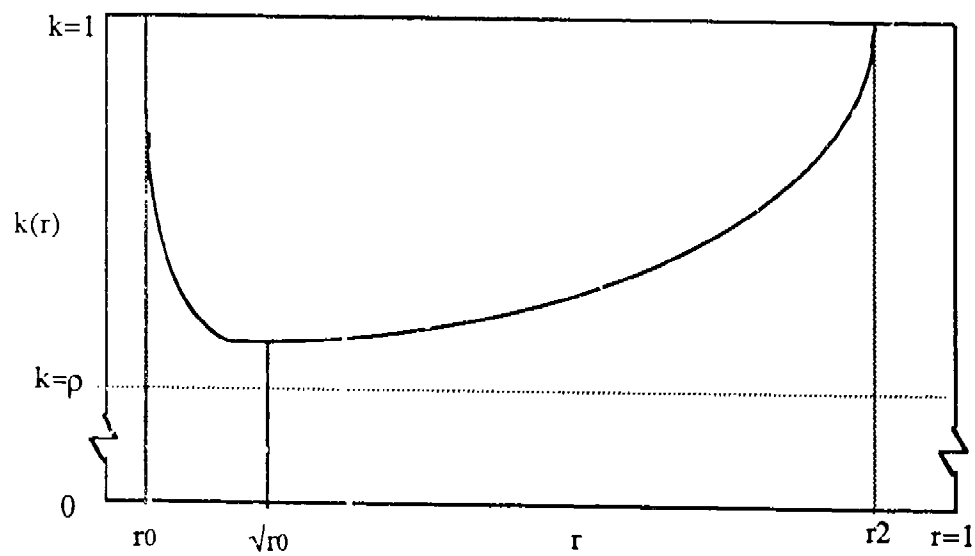

Figure 4: The relation between k(r) and r.

From equation (4.5.2), the right hand side limit of g'(r) diverges as r approaches r0 i.e.,

$$\lim_{r \to r0^+} g'(r) = +\infty. \tag{4.5.15}$$

So, the slope of k(r) at r = r0 is $-\infty$.

Note:

If we look at the above figure, we can see that the behavior of k(r) is very strange for r in $[r0, r_n]$. The reason is that we intuitively expect to have small k(r) when r is small. However, for r in this interval, k(r) grows as r decreases. Using larger k(r) means that a larger stopping time sequence $\{C_i(k)\}$ is used. So, this rule tells us to eliminate more defective items when we start with fewer for r in this range.

When the given range for r, say $[r_l, r_u]$, falls in (r0,r2), i.e.,

$r0 < r_l < r_u < r2,$

we obtain a conservative rule by taking

$k = \max\{k(r_l), k(r_u)\}.$

The reason is that the sequence of the stopping times $\{C_i(k)$, for $i=1,2,\ldots,n-1\}$ is an increasing function in k. In addition, we observe that $k(r0) = 1$, but unity is not a feasible value for k. We shall see in the next section that r must be bounded away from r0.

§ 4.6 How close to r_0 can r be?

The derivation of the formula for k(r) as a function of n in Section 4.4 depends on the assumption that burn-in can not be stopped at any stage j(m) where $(j(m)/m^{\mu}) \to 0$ $(1/2< \mu <1)$. The derivation also depends on studying the probability of stopping in a neighbourhood of $m_0 = (m_0(r,m))$ defined by (1.7.3). Thus, if $(m_0(r,m)/m^{\mu}) \to 0$ $(1/2 < \mu < 1)$, there is an inherent contradiction in the derivation. It is obvious from (1.7.3) that, as $r \to r_0^+$, $m_0(r,m) \to 0$. Hence, we must bound r away from r_0 in order for formula (4.4.5) to be valid. We shall obtain a lower bound $r^*(n) > r_0$ such that for $r > r^*(n)$, (4.4.5) is correct. For $r_0 < r < r^*(n)$, some formula for k(r) other than that given in this thesis must be used. We shall also find $k(r^*(n))$. This value will serve as an upper bound on k(r) for $r^*(n) \leq r < r_n$ as seen from the discussion of Section 4.5.

First of all, define

$f(r) = (r/(1-r))\cdot(1/b^2)\cdot((\rho-\exp(-t))/(1-\rho))$,

$g(r) = (r\cdot(1-\exp(-t))-(1-\rho))/(\rho-\exp(-t))$,

$e(r) = f(r)\cdot g(r)$,

$m = n\cdot r$,

$m_0(r,k) = (m\cdot(1-\exp(-t))-n\cdot(1-k))/(k-\exp(-t))$,

$k(r) = \rho + [r\cdot(\rho-\exp(-t))/(1-r)]\cdot\{b/[b+\sqrt{((n\cdot r)\cdot(r\cdot(1-\exp(-t))-(1-\rho))/[(1-\rho)\cdot(1-r)])}]\}$

(as given in (4.4.11).

So, we have

$1-r_0 = (\rho-\exp(-t))/(1-\exp(-t))$,

$r_0/(1-r_0) = (1-\rho)/(\rho-\exp(-t))$,

$m0(r,k(r)) \sim n\cdot\{g(r) - (r/(1-r))\cdot(1/\sqrt{(n\cdot e)})\cdot(1+g)\}$.

Consider the case $m0(r,k(r)) = a\cdot n^p$, where a is a constant and $1/2 \le p < 2/3$. We have

$(n^{1-p})\cdot\{g(r) - (r/(1-r))\cdot[(1+g)/(\sqrt{f(r)}\cdot g(r))]\cdot(1/\sqrt{n})\} = a$, or

$$(n^{1-p})\cdot g(r)\cdot\{1 - (r/(1-r))\cdot[(1+g)/(f^{1/2}(r)\cdot g^{3/2}(r))]\cdot(1/\sqrt{n})\} = a. \tag{4.6.1}$$

If n is large, we can see that

$$g(r) \sim A/(n^{(1/3)}) + B/(n^u) \tag{4.6.2}$$

where A and B are two constants, and $u > 1/3$.

Replacing g(r) in (4.5.1) by $A/(n^{(1/3)}) + B/(n^u)$, we have

$A\cdot(n^{1-p-1/3})\cdot\{1-(r/(1-r))\cdot(1/\sqrt{f})\cdot(1/A^{(3/2)})\cdot[1 - B/(A\cdot n^{(u-(1/3))})]^{(3/2)}-ç\cdot n^{(1/6)}/n^{(3/6)}\} = a$,

where ç is a constant. (4.6.3)

Since a is finite, we need the coefficient of the leading term to be zero, i.e.,

$$1-(r/(1-r))\cdot(1/\sqrt{f})\cdot(1/A^{(3/2)}) = 0 \text{ and} \tag{4.6.4}$$

$$u = 1-p, \tag{4.6.5}$$

since we need the power of n for the second term, namely $1-p-1/3-u+1/3$, equal to 0 .

Using (4.6.4), we have

$(r/(1-r))^{(1/2)}\cdot b\cdot[(1-\rho)/(\rho-\exp(-t))]^{(1/2)}\cdot(1/A^{(3/2)}) = 1$,

and using (4.6.12), we get

$b/A^{(3/2)} = 1$, or

$$A = b^{(2/3)}. \tag{4.6.6}$$

Using $g(r) = A/(n^{(1/3)}) + B/(n^{(1-p)})$ in (4.6.1), we obtain

$[A\cdot(n^{1-p-1/3})+B]\cdot\{(3/2)\cdot(r/(1-r))\cdot(1/\sqrt{f})\cdot(B/(A^{(5/2)}\cdot n^{(1-p-1/3)}) + \cdots\} = a$.

So,

$$(3/2)\cdot(r/(1-r))\cdot(1/\sqrt{f})\cdot(B/(A^{(3/2)}) \sim a\ , \tag{4.6.7}$$

We know that (4.6.7) is true if $1/2 \le p < 2/3$.

Using (4.6.4) to solve (4.6.7), we have

$$B = (2/3)\bullet a. \tag{4.6.8}$$

Let

$$r = r_0 + \partial. \tag{4.6.9}$$

We have

$$g(r) = g(r_0 + \partial) = (r_0\bullet(1-\exp(-t))-(1-\rho))/(\rho-\exp(-t)) + \partial\bullet(1-\exp(-t))/(\rho-\exp(-t))$$

$$= \partial\bullet(1-\exp(-t))/(\rho-\exp(-t))$$

$$= A/(n^{(1/3)}) + B/(n^{1-p}). \tag{4.6.10}$$

In addtion,

$$r/(1-r) = (r_0+\partial)/[(1-r_0)\bullet(1-(\partial/(1-r_0))]$$

$$\sim [(r_0+\partial)/(1-r_0)]\bullet(1+(\partial/(1-r_0))$$

$$= [(r_0+\partial)\bullet(1+(\partial/(1-r_0))]/(1-r_0)\}$$

$$= r_0/(1-r_0) + \partial/(1-r_0)^2$$

$$= (r_0/(1-r_0))\bullet(1 + \partial/(1-r_0)) \text{ and} \tag{4.6.11}$$

$$\sqrt{(r/(1-r))} \sim \sqrt{(r_0/(1-r_0))} + (1/2)\bullet\partial/(r_0\bullet(1-r_0))$$

$$\sim \sqrt{(r_0/(1-r_0))}$$

$$= \sqrt{[(1-\rho)/(\rho-\exp(-t))]} \tag{4.6.12}$$

From (4.6.6), (4.6.8) and (4.6.12), we get

$$\partial_{min} \sim [b^{(2/3)}/(n^{(1/3)}) + (2\bullet a/3)/(n^{1-p})]/[(1-\rho)/(\rho-\exp(-t))], \text{ let} \tag{4.6.13}$$

$$= C/(n^{(1/3)})+D/(n^{1-p}), \text{ for appropriate C and D.} \tag{4.6.14}$$

A effective lower bound of r is obtained:

$$r^*(n) = r_0 + \partial_{min}$$

$$= (1-\rho)/(\rho-\exp(-t)) + [b^{(2/3)}/(n^{(1/3)}) + (2\bullet a/3)/(n^{1-p})]/$$

$$[(1-\rho)/(\rho-\exp(-t))]. \tag{4.6.15}$$

What is k(r*(n))?

$$k(r^*(n)) = \rho + [r_1^* \cdot (\rho - \exp(-t))/(1-r_1^*)] \cdot \{b/[b+((n \cdot r_1^*) \cdot (r \cdot (1-\exp(-t))-(1-\rho)) / [(1-\rho) \cdot (1-r_1^*)])^{1/2}]\}$$

$$= \rho + (\rho - \exp(-t)) \cdot \{[(1-\rho)/(\rho-\exp(-t))]+[\partial \cdot (1-\exp(-t)^2/(\rho-\exp(-t))^2]\} \cdot b/\{b+\sqrt{[n \cdot (r0/(1-r0)) \cdot (1+\partial/(r0 \cdot (1-r0))) \cdot (1/(1-\rho)) \cdot (\partial \cdot (1-\exp(-t)))]}$$

$$= \rho + \{[(1-\rho)+[\partial \cdot (1-\exp(-t)^2/(\rho-\exp(-t))]\} \cdot 1/\{1+\sqrt{[n \cdot \partial \cdot (1+(\partial/(r0 \cdot (1-r0)))) \cdot (1-\exp(-t))/(b^2 \cdot (\rho-\exp(-t)))]}$$

$$\sim \rho + \{[(1-\rho)+[\partial \cdot (1-\exp(-t)^2/(\rho-\exp(-t))]\} \cdot [b \cdot (\sqrt{(\rho-\exp(-t))})/(\sqrt{(1-\exp(-t))} \cdot (\sqrt{C}) \cdot n^{(1/3)}] \cdot [1-D/(2 \cdot n^{(2/3-p)})]$$

$$\sim \rho + \{[(1-\rho)+[C/(n^{(1/3)})+D/(n^{1-p})] \cdot (1-\exp(-t)^2/(\rho-\exp(-t))]\} \cdot \{b \cdot [(\sqrt{(\rho-\exp(-t))})/(\sqrt{(1-\exp(-t))}]/[(\sqrt{C}) \cdot n^{(1/3)}]\}$$

$$\sim \rho + (1-\rho) \cdot b \cdot (\sqrt{(\rho-\exp(-t))})/\{(\sqrt{(1-\exp(-t))}) \cdot (\sqrt{C}) \cdot n^{(1/3)}\} \qquad (4.6.16)$$

This, (4.6.16), is a decreasing function in C, i.e., k(r*(n)) decreases as r*(n) increases.

So, if $r^*(n) < r < r_n$ we can use (4.6.16),

$$k = k(r^*(n)) = \rho + (1-\rho) \cdot b \cdot (\sqrt{(\rho-\exp(-t))})/\{(\sqrt{(1-\exp(-t))}) \cdot (\sqrt{C}) \cdot n^{(1/3)}\}$$

$$= \rho + (1-\rho) \cdot b \cdot (\sqrt{(\rho-\exp(-t))})/\{(\sqrt{(1-\exp(-t))}) \cdot (b^{(1/3)}) \cdot n^{(1/3)}\}. \qquad (4.6.17)$$

So, use k=max(k(r*(n)), k(r_u)) where r_u is the upper bound of r to obtain a conservative rule.

Summarizing the above discussion, we have the following theorem.

Theorem 4.6.1:

Suppose $\partial = [b^{(2/3)}/(n^{(1/3)}) + (2 \cdot a/3)/(n^{1-p})]/[(1-\rho)/(\rho-\exp(-t))]$ for some positive number a and $1/2 < p < 2/3$, and r_u is an upper bound of r. If $r^*(n) < r < r_u$ then $k=\max(k(r^*(n)), k(r_u))$ will give us a conservative rule, with $k(r^*(n))$ given by (4.6.17).

§4.7 Limiting Probability of Early Stopping.

Lemma 4.3.2 tells us that, for stopping rule (S.3.0), whether $P(R(t;D,m,n) \geq \rho) \geq \alpha$ is true or not mainly depends on the first several observed failure times of the defective items if $k > \rho$ and m is large enough. In other words,

$\text{limit}_{n\to\infty} P(R(t;D,m,n) \geq \rho) \geq \alpha$

depends mainly on

$\text{limit}_{m\to\infty} P(U_{1:m} \leq C_1(k), \cdots, U_{i:m} \leq C_i(k))$ for $i = 1,2, \cdots, j(m)$ (4.7.1)

where $j(m)/m^{\mu} \to 0$ and $1/2 < \mu \leq 1$.

Let's evaluate the case $j(m) = 1$ first. In §3 8, we studied $P(U_{1:m} \leq C_1(k))$ for a given m.

$\text{limit}_{n\to\infty} P(U_{1:m} \leq C_1(k))$

$= \text{limit}_{n\to\infty}\, 1 - \{1 - (1-\exp(-t))/[(n-1)\cdot(1-k)+(1-\exp(-t))]\}^m$

$= \text{limit}_{n\to\infty}\, 1 - \{1 - (1/m)\cdot(1-\exp(-t))/[(n-1)\cdot(1-k)/m+(1-\exp(-t))/m]\}^m$

$= 1 - \exp\{- r\cdot(1-\exp(-t))/(1-k)\}$,

where $r = \text{limit}_{n\to\infty} m/n$ and $0 \leq r < 1$.

Define

$g(r,k) = 1 - \exp\{-r\cdot(1-\exp(-t))/(1-k)\}$ (4.7.3)

We have the following lemma.

Lemma 4.7.1:

1. If burn-in is required (and n is large enough), then

   $g(r,k) \geq \text{limit}_{n\to\infty} P(R(t;D,m,n) \geq \rho)$.

2. If r is fixed, then

   $g(r,k)$ is an increasing function of k with $[1-\exp(-r\cdot(1-\exp(-t))),1]$ as its range.

3. If k is fixed, then g(r,k) is an increasing function of r.

Note: This is a trivial lemma and the proof of it is omitted.

For j=2,

$\text{limit}_{n->\infty} P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots,j)$

$= \text{limit}_{n->\infty} \{1-(1-C_1(k))\}^{m} - m\cdot C_1(k)\cdot\{1-C_2(k)\}^{m-1}$

$= 1-\exp\{-r\cdot(1-\exp(-t))/(1-k)\}-\{r\cdot(1-\exp(-t))/(1-k)\}\cdot\exp\{-2\cdot r\cdot(1-\exp(-t))/(1-k)\}$ (4.7.4)

For j=3,

$\text{limit}_{n->\infty} P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots,j)$

$=\text{limit}_{n->\infty} 1 - (1-C_1(k))\}^{m}-m\cdot C_1(k)\cdot\{1-C_2(k)\}^{m-1} - m\cdot(m-1)\cdot(1-C_3(k))^{m-2}\cdot\{C_2(k)\cdot C_1(k)-(1/2)\cdot C_1(k)^2\}$

$= 1- \exp\{-r\cdot(1-\exp(-t))/(1-k)\}-\{r\cdot(1-\exp(-t))/(1-k)\}\cdot\exp\{-2\cdot r\cdot(1-\exp(-t))/(1-k)\} -(3/2)\cdot\{r\cdot(1-\exp(-t))/(1-k)\}^2\cdot \exp\{-3\cdot r\cdot(1-\exp(-t))/(1-k)\}$ (4.7.5)

For j=4,

$\text{limit}_{n->\infty} P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots,j)$

$= \text{limit}_{n->\infty} 1 - (1-C_1(k))\}^{m}-m\cdot C_1(k)\cdot\{1-C_2(k)\}^{m-1} - m\cdot(m-1)\cdot(1-C_3(k))^{m-2}\cdot\{C_2(k)\cdot C_1(k)-(1/2)\cdot C_1(k)^2\}-m\cdot(m-1)\cdot(m-2)\cdot\{[C_1(k)\cdot C_2(k)\cdot C_3(k)-(1/2)\cdot C_2(k)\cdot C_1(k)^2] - (1/2)\cdot[C_2(k)_2\cdot C_1(k) - (1/3)\cdot C_1(k)^3]\}$

$= 1 - \exp\{-r\cdot(1-\exp(-t))/(1-k)\}-\{r\cdot(1-\exp(-t))/(1-k)\}\cdot\exp\{-2\cdot r\cdot(1-\exp(-t))/(1-k)\}-(3/2)\cdot\{r\cdot(1-\exp(-t))/(1-k)\}^2\cdot\exp\{-3\cdot r\cdot(1-\exp(-t))/(1-k)\}-(19/6)\cdot\{r\cdot(1-\exp(-t))/(1-k)\}^3\cdot\exp\{-4\cdot r\cdot(1-\exp(-t))/(1-k)\}.$ (4.7.6)

Define

$A = r\cdot(1-\exp(-t))/(1-k).$ (4.7.7)

Summarizing the above results, we have

$\text{limit}_{n\to\infty} P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots)$

$= 1 - \exp(-A) - A\cdot\exp(-2\cdot A) - (3/2)\cdot A^2\cdot\exp(-3\cdot A) - (19/6)\cdot A^3\cdot\exp(-4\cdot A) - \cdots$

$= 1 - a(1)\cdot\exp(-A) - a(2)\cdot A\cdot\exp(-2\cdot A) - a(3)\cdot(3/2)\cdot A^2\cdot\exp(-3\cdot A) - a(4)\cdot(19/6)\cdot A^3\cdot \exp(-4\cdot A) - \cdots$

$< 1 - \exp(-A) - A\cdot\exp(-2\cdot A) - A^2\cdot\exp(-3\cdot A) - A^3\cdot\exp(-4\cdot A) - \cdots$

$= 1 - \exp(-A)\cdot\{1 + A\cdot\exp(-A) + (A\cdot\exp(-A))^2 + (A\cdot\exp(-A))^3 - \cdots\}$

$= 1 - \exp(-A)/[1 - A\cdot\exp(-A)]$. (4.7.8)

This is an upper bound of $\text{limit}_{n\to\infty}P(R(t;D,m,n) \geq \rho)$ if we allow this burn-in process to stop at the very beginning. On the other hand, we conjecture that we should be able to find a constant $1 < ç < \infty$ such that

$a(i) \leq ç^i$ for $i = 1,2,3,\cdots, m^\mu$ with $1/2 < \mu < 1$ and

$\text{limit}_{n\to\infty}P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots,m^\mu)$

$\geq 1 - ç\cdot\exp(-A)/[1- ç\cdot A\cdot\exp(-A)]$. (4.7.9)

This is a lower bound of $\text{limit}_{n\to\infty}P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots,m^\mu)$.

If the right hand side of inequality (4.7.9) is positive for some k in (0,1), we should be able to find a suitable k, say k*, such that

the right hand side of (4.7.8) $\geq \alpha$. (4.7.10)

In this case, it is clear that

$\text{limit}_{n\to\infty} P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots,j+1)$

$= \text{limit}_{n\to\infty}P(U_{i:m}<C_i(k) \text{ for } i=1,2,\cdots,j) -$

$a(j)\cdot\{r\cdot(1-\exp(-t))/(1-k)\}^{j-1}\cdot\exp\{-j\cdot r\cdot(1-\exp(-t))/(1-k)\}$, (4.7.11)

And, we can conclude that

$\Sigma_{j\geq 0}\, a(j)\cdot\{r\cdot(1-\exp(-t))/(1-k)\}^{j-1}\cdot\exp\{-j\cdot r\cdot(1-\exp(-t))/(1-k)\}$ (4.7.12) is close to 0 for moderate ∂. That is, for any error bound e, there exists a moderate ∂ such that

$\text{limit}_{n\to\infty} P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots,\partial)$

$- \text{limit}_{n\to\infty} P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots,m^{\mu}) \le e$ (4.7.13)

In addition, if $k > \rho$, we'll have

$\text{limit}_{n\to\infty} P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots,\partial)$

$\ge \text{limit}_{n\to\infty} P(U_{i:m} < C_i(k) \text{ for } i=1,2,\cdots,m^{\mu})$

$= \text{limit}_{n\to\infty} P(R(t;D,m,n) \ge \rho)$ (4.7.14)

The following argument shows $k > \rho$ when k is derived from early failures. This tells us that the derivation of k in this section is not close to ρ no matter how large n is, i.e., the stopping rule obtained by using this k will be too conservative. From (4.7.8), the upper bound on the confidence is given by

$U^b(A) = (1-\exp(-A))/(1-A\bullet\exp(-A))$ where A is given by (4.7.7).

Note that the first derivative of $U^b(A)$ is

$U^{b'}(A) = \{(\exp(-A))\bullet(1-A\bullet\exp(-A)) + (1-\exp(-A))\bullet(\exp(-A))\bullet(1-A)\}/(1-A\bullet\exp(-A))^2$

$\quad = \{(\exp(-A))\bullet[1-A\bullet\exp(-A) + 1-\exp(-A)-A + A\bullet\exp(-A)]\}/(1-A\bullet\exp(-A))^2$

$\quad = \{(\exp(-A))\bullet[2 - (A+\exp(-A))]\}/(1-A\bullet\exp(-A))^2.$ (4.7.15)

$(d/dA)[A+\exp(-A)] = 1 - \exp(-A).$

So, $A+\exp(-A)$ is an increasing function in A. In addition,

$U^{b'}(A) > 0$ for $A < A^*$ where $A^*+\exp(-A^*) = 2$ ($A^* < 2$ but near 2), and

$U^{b'}(A) < 0$ for $A > A^*$. (4.7.16)

If $A < 1$, then $A\bullet\exp(-A) < \exp(-A)$ or $1 - A\bullet\exp(-A) > 1-\exp(-A)$. So,

$U^b(A) < 1$ if $A < 1$. (4.7.17)

Furthermore,

$U^b(1) = 1.$ (4.7.18)

Hence, there is a solution $A1 < 1$ of $U^b(A) = \alpha$. (4.7.19)

Also a solution $A2 > 2$ of $U^b(A) = \alpha$. (4.7.20)

In the range of interest, $r > (1-\rho)/(1-\exp(-t))$ or

$A > (1-\rho)/(1-k)$. (4.7.21)

Note: $k \geq \rho \iff A \geq 1$. (4.7.22)

So only solution $A2 > 2$ of $U^b(A2) = \alpha$ is of interest.

For any r in the range of interest $r > r0$, define $r = e \cdot r0$ with $e > 1$. We have

$1 - k = (r/A2) \cdot (1-\exp(-t)) = e \cdot (1-\rho)/A2.$ (4.7.23)

So, k decreases as r (or e) increases and a conservative k is that for the minimum r. Suppose the minimum r is near r0 and i.e., $e \sim 1$, then from (4.7.22), we see that $k > \rho$. If $P(R(t;D,m,n) \geq \rho)$ increases with k, then we need $k > k'$ where k' is given by (4.7.23), and $k' > \rho$ when e is less then A2.

§4.8 Use $k = \rho + O(1/\sqrt{n})$

Let k1 be the "k" obtained from (4.4.5) and k2 be the "k" derived by equating (4.7.12) to α, if a solution exists. The comparison about the performances of the burn-in processes based on k1 and k2 is given and k1 is recommended as the better choice.

Let's consider the case that k2 is used. From Section 4.7, we know that $k2 > \rho$. In this case, we will have about $100\cdot(1-\alpha)\%$ chance of having a lot with most of its defective items still remaining after burn-in and have about $100\cdot\alpha\%$ chance of having a lot with many fewer defective items than (m-mo). So, if k2 is used, the quality of any lot after burn-in has two extremes: this lot can be very bad with most of its defective items still sitting there and the duration of this burn-in is very short; or this lot can be too good with many fewer defective items in it and with its duration of burning-in, $-\ln(1-s_0(k2))$, longer than what is required. Although we might still achieve our reliability goal if k2 is used, the over all quality of the lots after burning-in is not consistent and the duration of burning-in would tend to longer than it truly needs to be, i.e., $(s_0(k2) > s_0(\rho)$ if $k2 > \rho$.

If k1 is used, burn-in will be stopped approximately when mo(k1) defective items are removed from a burn-in lot. The quality of each after-burn-in lot is very similar. For a lot put on burn-in, the number of the defective items remaining in it is close to m-mo(k) after burning-in. In this case, each burn-in process also tends to stop at the same time $-\ln(1-s_0(k1))$. In addition, $s_0(k1)$ is getting close to $s_0(\rho)$, the lower bound of $s_0(k)$, if m is getting larger.

Based on the consideration about the consistency of the quality of each after-burn-in lot and the possible duration of burn-in, the k derived by using (4.4.5) is recommended.

CHAPTER V

NUMERICAL RESULTS, SIMULATIONS, AND COMPARISONS

§5.1 Introduction

In this chapter, the achieved confidence level and the expected duration are computed (Procedure 0 and Procedure I) or simulated (Procedure II) for each procedure. The same true m and a set of assumed "m"s are used for each procedure, too. So, we can see how sensitive the confidence and the duration of burn-in are to the assumed value of m. In addition, we can make comparisons among these procedures.

For each procedure, two sets of (n,m) values are used. One is for a small lot where n equals 400 and m ranges from 32 to 352 in steps of 32, and the other one is for a large lot size where n equals 4000 and m ranges from 320 to 3520 in steps of 320. In this chapter, we use ρ=.99, α=.99, and 1-exp(-t)=.99, i.e., t=4.61 for all the numerical computations and the simulation runs. For Procedure II, 2000 simulations were run. We can compare the results in each table, and see the differences among these procedures and see the differences in each individual procedure when different (assumed or true) numbers of defective items, m, or burn-in lot size, n, are used.

§ 5.2 The Numerical Results for Procedure 0

The tables in this section show the numerical evaluation of the performance of Procedure 0. Tables 1 to 5 tell us the performance of Procedure 0 when the lot size n=400 with the assumed (or true) number of defective items m ranging from 64 to 352. Tables 6 to 10 tell us the performance of this procedure when the lot size n is 4000 with m ranging from 640 to 3520.

Notations used in tables:

n: the burn-in lot size.

m: the true number of the defective items in a burn-in lot.

mest = m^{est}: the assumed number of the defective items in this burn-in lot.

m0 = m_0: this number of defective items must be eliminated through burn-in to achieve our reliability goal.

mest0 = m^{est}_0: this is the number of defective items which we intend to eliminate through burn-in to achieve our reliability goal when the assumed m is mest.

t: this is the required after burn-in service period.

t-delta: this is the stopping time of Procedure 0. It is denoted as Δ in Chapter 0.

confidence: this is the probability of achieving $R(t;D,m,n)\geq\rho$, $P(R(t;D,m,n)\geq\rho)$, when this stopping rule uses the stopping time which is derived by an assumed value of m.

If we look at Table 1 to Table 5, we can see that 'confidence' increases steadily as the assumed value of m increases. If m is under estimated, we will not be able to achieve $R(t;D,m,n)\geq\rho$ with the desired probability α, i.e. we will be unable to achieve our reliability goal. The chance of achieving our reliability goal is getting worse if the assumed value of m is farther below the true value of m. If m is over estimated, the quality of a lot after burn-in is higher than the required quality with a longer duration of burn-in.

If we compute the difference m - m_0, we have that m-m_0 is a decreasing function of m. This tells us that fewer defective items can remain after burn-in if we have more defective items in a given burn-in lot. Here, m^{est}-m^{est}_0 = 0 if m^{est} is greater than or equal to 320. The difference m^{est}-m^{est}_0 has a great influence on the stopping times. If we check the column of t-delta, we can see that t-delta increases steadily as the assumed value, m^{est}, increases and has a jump when the value of the difference m^{est}-m^{est}_0 changes, for example, when m^{est} goes from 96 to 128
m^{est}-m^{est}_0 changes from three to two. In addition, the stopping time t-delta has a great jump from m^{est}=298 to m^{est}=320. The reason for this jump is that the difference m^{est}-m^{est}_0 changes from 1 to 0.

If we look at Table 6 to Table 10 (the large lot size, n=4000, case), we can see results similar to those described in the previous paragraph. In addition to these, we can see another two features in these tables. The first one is that 'confidence', the chance for us to achieve R(t;D,m,n)≥ρ, is very sensitive to the assumed value of m if it is under estimated. The second one is that t-delta is significantly smaller than the t-delta in the first five tables. This tells us the fact that the duration of burn-in will be reduced if the lot size is increased for a fixed ratio (m/n).

Table 1: n=400, m=64, mo=61

mest	mest0	t-delta	confidence
32	29	3.62492	0.908605
64	61	4.33606	0.990000
96	93	4.74737	0.997622
128	126	5.67589	0.999926
160	158	5.90027	0.999969
192	190	6.08341	0.999985
224	223	7.31654	1.000000
256	255	7.45031	1.000000
288	287	7.56828	1.000000
320	320	10.36849	1.000000
352	352	10.46379	1.000000

Table 2: n=400, m=128, mo=126

mest	mest0	t-delta	confidence
32	29	3.62492	0.333892
64	61	4.33606	0.764306
96	93	4.74737	0.899116
128	126	5.67589	0.990000
160	158	5.90027	0.994565
192	190	6.08341	0.996727
224	223	7.31654	0.999906
256	255	7.45031	0.999937
288	287	7.56828	0.999955
320	320	10.36849	1.000000
352	352	10.46379	1.000000

Table 3: n=400, m=192, mo=190

mest	mest0	t-delta	confidence
32	29	3.62492	0.111856
64	61	4.33606	0.539644
96	93	4.74737	0.766685
128	126	5.67589	0.971015
160	158	5.90027	0.983743
192	190	6.08341	0.990000
224	223	7.31654	0.999690
256	255	7.45031	0.999790
288	287	7.56828	0.999851
320	320	10.36849	1.000000
352	352	10.46379	1.000000

Table 4: n=400, m=256, m_0=255

mest	mest0	t-delta	confidence
32	29	3.62492	0.007949
64	61	4.33606	0.150741
96	93	4.74737	0.348278
128	126	5.67589	0.780792
160	158	5.90027	0.843953
192	190	6.08341	0.883566
224	223	7.31654	0.987113
256	255	7.45031	0.990000
288	287	7.56828	0.992016
320	320	10.36849	0.999968
352	352	10.46379	0.999974

Table 5: n=400, m=320, m_0=320

mest	mest0	t-delta	confidence
32	29	3.62492	0.000176
64	61	4.33606	0.014760
96	93	4.74737	0.061547
128	126	5.67589	0.333297
160	158	5.90027	0.415785
192	190	6.08341	0.481642
224	223	7.31654	0.808402
256	255	7.45031	0.830226
288	287	7.56828	0.847599
320	320	10.36849	0.990000
352	352	10.46379	0.990905

Table 6: n=4000, m=640, m0=606

mest	mest0	t-delta	confidence
320	283	2.52456	0.005219
640	606	3.32923	0.990000
960	929	3.84950	0.999999
1280	1253	4.30731	1.000000
1600	1576	4.67666	1.000000
1920	1899	5.02625	1.000000
2240	2223	5.44912	1.000000
2560	2546	5.83469	1.000000
2880	2869	6.27297	1.000000
3200	3192	6.81531	1.000000
3520	3516	7.91967	1.000000

Table 7: n=4000, m=1280, m0=1253

mest	mest0	t-delta	confidence
320	283	2.52456	0.000000
640	606	3.32923	0.001586
960	929	3.84950	0.531488
1280	1253	4.30731	0.990000
1600	1576	4.67666	0.999955
1920	1899	5.02625	1.000000
2240	2223	5.44912	1.000000
2560	2546	5.83469	1.000000
2880	2869	6.27297	1.000000
3200	3192	6.81531	1.000000
3520	3516	7.91967	1.000000

Table 8: n=4000, m=1920, m0=1899

mest	mest0	t-delta	confidence
320	283	2.52456	0.000000
640	606	3.32923	0.000000
960	929	3.84950	0.000423
1280	1253	4.30731	0.195984
1600	1576	4.67666	0.808547
1920	1899	5.02625	0.990000
2240	2223	5.44912	0.999950
2560	2546	5.83469	1.000000
2880	2869	6.27297	1.000000
3200	3192	6.81531	1.000000
3520	3516	7.91967	1.000000

Table 9: n=4000, m=2560, m_0=2546

mest	mest0	t-delta	confidence
320	283	2.52456	0.000000
640	606	3.32923	0.000000
960	929	3.84950	0.000000
1280	1253	4.30731	0.000051
1600	1576	4.67666	0.020948
1920	1899	5.02625	0.296151
2240	2223	5.44912	0.853912
2560	2546	5.83469	0.990000
2880	2869	6.27297	0.999846
3200	3192	6.81531	1.000000
3520	3516	7.91967	1.000000

Table 10: n=4000, m=3200, m_0=3192

mest	mest0	t-delta	confidence
320	283	2.52456	0.000000
640	606	3.32923	0.000000
960	929	3.84950	0.000000
1280	1253	4.30731	0.000000
1600	1576	4.67666	0.000002
1920	1899	5.02625	0.001074
2240	2223	5.44912	0.069317
2560	2546	5.83469	0.409274
2880	2869	6.27297	0.843571
3200	3192	6.81531	0.990000
3520	3516	7.91967	0.999996

§ 5.3 The Numerical Results for Procedure I

We numerically evaluate the performance of Procedure I in this section. Similarly to the previous section, the first five tables evaluate the performance of this procedure when the lot size is small, i.e., n=400. The last table tells us the performance of this procedure when the lot size is large. As mentioned in Section 2.3, we use three equations, (2.3.3), (2.3.4) and (2.3.5), to compute upper bounds t*, for the waiting times between successive failures, so the tables in this section have all these three t*s computed. The three corresponding expected durations of burn-in time are also computed here.

Notations used in these tables:

(The notations which were defined in §5.2 will not be given here.)

t*(3): The upper bound, which is calculated by using equation (2.3.3), for the waiting times between successive failures.

t*(4): The upper bound, which is calculated by using equation (2.3.4), for the waiting times between successive failures.

t*(5): The upper bound, which is calculated by using equation (2.3.5), for the waiting times between successive failures.

ED(3): The expected duration of burn-in of Procedure I when upper bound t*(3) is used.

ED(4): The expected duration of burn-in of Procedure I when upper bound t*(4) is used.

ED(5): The expected duration of burn-in of Procedure I when upper bound t*(5) is used.

$P(R(3))$: The probability of achieving $R(t,D,m,n) \geq \rho$ when upper bound $t^*(3)$ is used.

$P(R(4))$: The probability of achieving $R(t,D,m,n) \geq \rho$ when upper bound $t^*(4)$ is used.

$P(R(5))$: The probability of achieving $R(t,D,m,n) \geq \rho$ when upper bound $t^*(5)$ is used.

For the small lot case, Table 11 to Table 15, we can see that t*s and EDs change only when $m^{est}-m^{est}{}_0$ is changed. There is no significant difference among $t^*(3)$, $t^*(4)$ and $t^*(5)$, nor among the corresponding expected durations. This tells us that we can use the most simplified equation to compute t^* without losing too much. If n and the true m are fixed, then the probability of achieving $R(t,D,m,n) \geq \rho$ increases as $m^{est}-m^{est}{}_0$ decreases and this probability does not have any change when m^{est} increases but $m^{est}-m^{est}{}_0$ remains constant. If the assumed m is less than the true m and $m^{est}-m^{est}{}_0$ is the same as $m-m_0$, then we will be able to achieve our reliability, $P(R(t,D,m,n)\geq\rho)\geq \alpha$. If the assumed m is less than the true m and $m^{est}-m^{est}{}_0$ is larger than $m-m_0$, then this reliability goal is not achievable. However, the drop-off in confidence as m^{est} falls below m is not nearly as great as with Procedure 0. If the value of m is over-estimated, the quality of an after-burn-in lot will be higher than what is required and the duration of burn-in will also be longer than what is needed.

For the large lots, n=4000, the numerical results tell us the same properties of this procedure as the small lot size case tells us. Similarly to the large lot case of Procedure 0, the expected duration of burn-in is significantly reduced if the lot size is increased from 400 to 4000 when the assumed value of m is not too close to n. (The performance of Procedure I when the lot size is large is similar to its performance when n is small, so the other tables given in Section 5.2 will not be given here.)

Table 11: n= 400, m = 64, m0=61

mest	mest0	t*(3)	t*(4)	t*(5)	ED(3)	ED(4)	ED(5)	P(R(3))	P(R(4))	P(R(5))
32	29	1.2364	1.2370	1.237	5.1762	5.1771	5.1771	.9900	.9900	.9900
64	61	1.2364	1.2370	1.237	5.1762	5.1771	5.1771	.9900	.9900	.9900
96	93	1.2364	1.2370	1.237	5.1762	5.1771	5.1771	.9900	.9900	.9900
128	126	1.6089	1.6094	1.609	5.7623	5.7630	5.7630	.9979	.9980	.9980
160	158	1.6089	1.6094	1.609	5.7623	5.7630	5.7630	.9979	.9980	.9980
192	190	1.6089	1.6094	1.609	5.7623	5.7630	5.7630	.9979	.9980	.9980
224	223	2.3521	2.3525	2.352	6.7626	6.7631	6.7631	.9999	.9999	.9999
256	255	2.3521	2.3525	2.352	6.7626	6.7631	6.7631	.9999	.9999	.9999
288	287	2.3521	2.3525	2.352	6.7626	6.7631	6.7631	.9999	.9999	.9999
320	320	4.6150	4.6151	4.615	9.3031	9.3032	9.3032	1.000	1.000	1.000
352	352	4.6150	4.6151	4.615	9.3031	9.3032	9.3032	1.000	1.000	1.000

Table 12: n=400, m=128, m0=126

mest	mest0	t*(3)	t*(4)	t*(5)	ED(3)	ED(4)	ED(5)	P(R(3))	P(R(4))	P(R(5))
32	29	1.2364	1.2370	1.237	5.865	5.8664	5.8664	.9557	.9658	.9658
64	61	1.2364	1.2370	1.237	5.865	5.8664	5.8664	.9657	.9658	.9658
96	93	1.2364	1.2370	1.237	5.865	5.8664	5.8664	.9657	.9658	.9658
128	126	1.6089	1.6094	1.609	6.451	6.4523	6.4523	.9900	.9900	.9900
160	158	1.6089	1.6094	1.609	6.451	6.4523	6.4523	.9900	.9900	.9900
192	190	1.6089	1.6094	1.609	6.451	6.4523	6.4523	.9900	.9900	.9900
224	223	2.3521	2.3525	2.352	7.451	7.4524	7.4524	.9990	.9990	.9990
256	255	2.3521	2.3525	2.352	7.451	7.4524	7.4524	.9990	.9990	.9990
288	287	2.3521	2.3525	2.352	7.451	7.4524	7.4524	.9990	.9990	.9990
320	320	4.6150	4.6151	4.615	9.992	9.9925	9.9925	.9999	.9999	.9999
352	352	4.6150	4.6151	4.615	9.992	9.9925	9.9925	.9999	.9999	.9999

Table 13: n=400, m=192, m0=190

mest	mest0	t*(3)	t*(4)	t*(5)	ED(3)	ED(4)	ED(5)	P(R(3))	P(R(4))	P(R(5))
32	29	1.2364	1.2370	1.237	6.269	6.2705	6.2705	.9657	.9658	.9658
64	61	1.2364	1.2370	1.237	6.269	6.2705	6.2705	.9657	.9658	.9658
96	93	1.2364	1.2370	1.237	6.269	6.2705	6.2705	.9657	.9658	.9658
128	126	1.6089	1.6094	1.609	6.855	6.8564	6.8564	.9900	.9900	.9900
160	158	1.6089	1.6094	1.609	6.855	6.8564	6.8564	.9900	.9900	.9900
192	190	1.6089	1.6094	1.609	6.855	6.8564	6.8564	.9900	.9900	.9900
224	223	2.3521	2.3525	2.352	7.856	7.8565	7.8565	.9990	.9990	.9990
256	255	2.3521	2.3525	2.352	7.856	7.8565	7.8565	.9990	.9990	.9990
288	287	2.3521	2.3525	2.352	7.856	7.8565	7 8565	.9990	.9990	.9990
320	320	4.6150	4.6151	4.615	10.396	10.3966	10.3966	.9999	.9999	.9999
352	352	4.6150	4.6151	4.615	10.396	10.3966	10.3966	.9999	.9999	.9999

Table 14: n=400, m=256, m_0=255

mest	mest0	t*(3)	t*(4)	t*(5)	ED(3)	ED(4)	ED(5)	P(R(3))	P(R(4))	P(R(5))
32	29	1.2364	1.2370	1.237	6.556	6.5576	6.5576	.8843	.8844	.8844
64	61	1.2364	1.2370	1.237	6.556	6.5576	6.5576	.8843	.8844	.8844
96	93	1.2364	1.2370	1.237	6.556	6.5576	6.5576	.8843	.8844	.8844
128	126	1.6089	1.6094	1.609	7.142	7.1435	7.1435	.9503	.9504	.9504
160	158	1.6089	1.6094	1.609	7.142	7.1435	7.1435	.9503	.9504	.9504
192	190	1.6089	1.6094	1.609	7.142	7.1435	7.1435	.9503	.9504	.9504
224	223	2.3521	2.3525	2.352	8.143	8.1436	8.1436	.9900	.9900	.9900
256	255	2.3521	2.3525	2.352	8.143	8.1436	8.1436	.9900	.9900	.9900
288	287	2.3521	2.3525	2.352	8.143	8.1436	8.1436	.9900	.9900	.9900
320	320	4.6150	4.6151	4.615	10.683	10.6837	10.6837	.9999	.9999	.9999
352	352	4.6150	4.6151	4.615	10.683	10.6837	10.6837	.9999	.9999	.9999

Table 15: n=400, m=320, m_0=320

mest	mest0	t*(3)	t*(4)	t*(5)	ED(3)	ED(4)	ED(5)	P(R(3))	P(R(4))	P(R(5))
32	29	1.2364	1.2370	1.237	6.7794	6.7803	6.7803	.6275	.6277	.6277
64	61	1.2364	1.2370	1.237	6.7794	6.7803	6.7803	.6275	.6277	.6277
96	93	1.2364	1.2370	1.237	6.7794	6.7803	6.7803	.6275	.6277	.6277
128	126	1.6089	1.6094	1.609	7.3655	7.3662	7.3662	.7601	.7603	.7603
160	158	1.6089	1.6094	1.609	7.3655	7.3662	7.3662	.7601	.7603	.7603
192	190	1.6089	1.6094	1.609	7.3655	7.3662	7.3662	.7601	.7603	.7603
224	223	2.3521	2.3525	2.352	8.3658	8.3663	8.3663	.8957	.8954	.8958
256	255	2.3521	2.3525	2.352	8.3658	8.3663	8.3663	.8957	.8958	.8958
288	287	2.3521	2.3525	2.352	8.3658	8.3663	8.3663	.8957	.8958	.8958
320	320	4.6150	4.6151	4.615	10.9063	10.9064	10.9064	.9900	.9900	.9900
352	352	4.6150	4.6151	4.615	10.9063	10.9064	10.9064	.9900	.9900	.9900

Table 16: n=4000, m=640, m0=606

mest	mest0	t*(3)	t*(4)	t*(5)	ED(3)	ED(4)	ED(5)	P(R(3))	P(R(4))	P(R(5))
320	283	0.1699	0.1700	0.1700	3.9356	3.9365	3.9365	.9834	.9834	.9834
640	606	0.1825	0.1827	0.1827	4.0441	4.0451	4.0451	.9900	.9900	.9900
960	929	0.1974	0.1976	0.1976	4.1540	4.1650	4.1650	.9944	.9944	.9944
1280	1253	0.2219	0.2221	0.2221	4.3460	4.3470	4.3470	.9978	.9978	.9978
1600	1576	0.2450	0.2452	0.2452	4.5038	4.5048	4.5048	.9991	.9991	.9991
1920	1899	0.2740	0.2742	0.2742	4.6857	4.6867	4.6867	.9997	.9997	.9997
2240	2223	0.3266	0.3268	0.3268	4.9798	4.9808	4.9808	.9999	.9999	.9999
2560	2546	0.3830	0.3833	0.3833	5.2558	5.2569	5.2569	.9999	.9999	.9999
2880	2869	0.4657	0.4660	0.4660	5.6038	5.6049	5.6049	1.000	1.000	1.000
3200	3192	0.5997	0.6001	0.6001	6.0661	6.0671	6.0671	1.000	1.000	1.000
3520	3516	1.0109	1.0114	1.0114	7.0606	7.0616	7.0616	1.000	1.000	1.000

Table 17: n=4000, m=1280, m0=1253

mest	mest0	t*(3)	t*(4)	t*(5)	ED(3)	ED(4)	ED(5)	P(R(3))	P(R(4))	P(R(5))
320	283	0.1699	0.1700	0.1700	4.6283	4.6292	4.6292	.9464	.9466	.9466
640	606	0.1825	0.1827	0.1827	4.7369	4.7378	4.7378	.9645	.9646	.9646
960	929	0.1974	0.1976	0.1976	4.8568	4.8577	4.8577	.9780	.9781	.9781
1280	1253	0.2219	0.2221	0.2221	5.0387	5.0397	5.0397	.9900	.9900	.9900
1600	1576	0.2450	0.2452	0.2452	5.1965	5.1975	5.1975	.9951	.9952	.9952
1920	1899	0.2740	0.2742	0.2742	5.3784	5.3795	5.3795	.9930	.9980	.9980
2240	2223	0.3266	0.3268	0.3268	5.6725	5.6736	5.6736	.9996	.9996	.9996
2560	2546	0.3830	0.3833	0.3833	5.9486	5.9497	5.9497	.9999	.9999	.9999
2880	2869	0.4657	0.4660	0.4660	6.2966	6.2977	6.2977	.9999	.9999	.9999
3200	3192	0.5997	0.6001	0.6001	6.7588	6.7599	6.7599	1.000	1.000	1.000
3520	3516	1.0109	1.0114	1.0114	7.7534	7.7543	7.7543	1.000	1.000	1.000

§ 5.4 The Numerical and Simulation Results for Procedure II

The performance of Procedure II is evaluated by simulation. Before this evaluation can be done, we must decide which k value should be used. In Section 4.4, a computation scheme to find a suitable value for k is derived by using equation (4.4.10). If the lot size is large, i.e., n=4000, this computation scheme gives us an excellent value of k (Table 24 and Table 25). However, the simulation study tells us that the value of k obtained by using Equation (4.4.10) is not applicable when the lot size, n, is small, i.e., n=400. In this case, the value of k calculated by using Equation (4.4.10) is above 1 very often.

When the lot size is small, the value of k is obtained by using a simulation. We pick a number between ρ and 1, and run a simulation to see how well this k performs, for a given m. If our reliability goal $P(R(t;D,m,n)\geq\rho) \geq \alpha$ is ensured by using this k, then we'll check whether the simulated probability value is close to α or not. If the simulated probability is not too far away from α, then we can use this k. Otherwise, we need to pick a smaller k ($\geq\rho$). On the other hand, if our simulated probability is smaller than α, then pick a larger k and do this simulation again, that is: repeat this procedure until an acceptable k is obtained. Table 18 to Table 24 tell us how k is obtained when the lot size n is 400.

Before we have any further discussions in this chapter, let's define the notation used in this section. (The notation defined in the previous two sections will not be repeated here.)

expected-dt: This is the expected duration of burn-in when this burn-in procedure is used.

k(mest): The value of k derived from Equation (4.4.10) with the assumed value of m being mest.

For the small lot size case, we vary the value of k from .9940 to .9995, i.e., k=.9940 is used in Table 18, k=.9950 is used for Table 19, ..., k=.9995 is used for Table 24. When the lot size is 400 and the true number of the defective items is 64, by comparing the "confidence" column of Tables 18 to 24, we can see that our reliability goal is achieved, if k=.996 (Table 20) is used. Similarly, when the lot size is 400 and the true number of the defective items is 128, comparing the "confidence" column of Tables 18 to 24, we can see that our reliability goal is achieved, if k=.998 (Table 22) is used. So, for the case of small lot size burn-in, we can use this approach to obtain the appropriate k.

Since we never know the true value of m, we design Procedure 2 by using the k corresponding to our estimated m. For instance, if we guess m to be 64, we use k=0.996 which was pointed out above as the appropriate choice in that case. Table 20 can now be read as showing how confidence and expected duration vary with different values of the true m when the estimated m is fixed at 64 (the only effect m^{est} has is in determining the value of k to use). This is different from the previous sections where each table showed confidence and expected duration as m^{est} varied for a fixed value of m. For instance, when $m^{est} = 64$, Table 20 shows that when m=352 (so that we have a major underestimate), the confidence is still 0.987. Table 20 also shows that if the true m is 32, the achieved confidence is 0.984 so that, for this procedure, overestimating m does not guarantee the confidence requirement. A look at Tables 18 to 24 suggests that the confidence of Procedure II is less sensitive than that of the other two to mis-specification of m.

For the large lot size case, the value of k can be derived from Equation (4.4.10). Table 25 tells us a simulation result for the case: m=640, n=4000, ρ=.99, 1-exp(t) =.99, α=.99 when the assumed value of m, m^{est}, is ranging from 320 to 3520 and the number of simulation runs is 2000. The expected duration of burn-in given in this table shows that it, expected-dt, increases as the assumed value of m is increases but it is not very significant for a very wide range of m (by comparing to the other procedures). Furthermore, the values in the "confidence" column or "k(m^{est})" column preserves the monotone property: their values increase as m increases. But, these are not always monotone. For "confidence", if we let the increment between the consecutive m^{est}'s be small enough, then we will be able to see that "confidence" is not a monotone function in m. This numerical result is not shown here. For k(m^{est}), we can see from Figure 4 of Section 4.6 that k(m) is a U-shaped function of r (or m). In addition, all the values in the "confidence" column are above the required lower bound α, i.e., our reliability goal is always ensured in this case: large lot size burn-in.

Table 26 shows a simulation result for the case n=4000, ρ=.99, 1-exp(-t)=.99, α=.99 when the assumed value of m is the same as the true value of m, where m varies from 320 to 3520. The number of simulation runs is 2000. We can see that the values in "confidence" column have a very little variation as in Table 24. In addition, all of these values in "confidence" column are above 0.99, so k(m^{est}) obtained here is conservative. Furthermore, the value in the "expected-dt" column increases as m increases, too. An application of this table is that, if an upper bound of m in a burn-in lot is known, it can be used to figure out the upper bound of the duration of burn-in which might be required to achieve the desired confidence For example, if the upper bound of m is 1600, n is 4000 and the desired confidence is 0.99, then the upper bound of the expected duration of burn-in will be around 4.81.

Table 18: k=0.9940, n=400

m	m0	confidence	expected-dt
32	29	0.917499900	2.688606739
64	61	0.948999941	3.461193562
96	93	0.960499883	3.960860491
128	126	0.916499913	4.357289791
160	158	0.936499894	4.704833984
192	190	0.964999914	5.029990196
224	223	0.900999904	5.351178169
256	255	0.946999967	5.686598301
288	287	0.975999951	6.056697369
320	320	0.916499972	6.502041340
352	352	0.963999927	7.116232872

number of simulation runs=2000

Table 19: k=0.9950, n=400

m	m0	confidence	expected-dt
32	29	0.952999890	2.872345686
64	61	0.971499920	3.644049644
96	93	0.977999866	4.144165039
128	126	0.948499918	4.540705204
160	158	0.959499955	4.888323784
192	190	0.979999900	5.213341713
224	223	0.924999952	5.534407616
256	255	0.963999927	5.869949341
288	287	0.983999908	6.240071774
320	320	0.939999938	6.685378551

number of simulation runs=2000

Table 20: k=0.9960, n=400

m	m0	confidence	expected-dt
32	29	0.984499872	3.230851650
64	61	0.992499948	4.001834869
96	93	0.995499909	4.502047062
128	126	0.977999926	4.898869514
160	158	0.986999869	5.246265888
192	190	0.990999937	5.571498394
224	223	0.973999916	5.892687798
256	255	0.984499931	6.227913857
288	287	0.990999937	6.598282337
320	320	0.966999948	7.043487072
352	352	0.986999929	7.657604694

number of simulation runs=2000

Table 21: k=0.9970, n=400

m	m0	confidence	expected-dt
32	29	0.992999911	3.385284662
64	61	0.995999932	4.156266212
96	93	0.997999907	4.657020092
128	126	0.986999929	5.053611755
160	158	0.991999924	5.400946140
192	190	0.993499935	5.725118088
224	223	0.985499918	6.047466278
256	255	0.990499914	6.382554531
288	287	0.994499922	6.753048420
320	320	0.975999951	7.198336124
352	352	0.990999937	7.812598705

number of simulation runs=2000

Table 22: k=0.9980, n=400

m	m0	confidence	expected-dt
32	29	0.997999907	3.792477846
64	61	0.998999894	4.562776566
96	93	0.999499917	5.063402653
128	126	0.996499956	5.460057259
160	158	0.996999919	5.807489395
192	190	0.999999940	6.132760525
224	223	0.995499909	6.453858852
256	255	0.996999919	6.788960457
288	287	0.998499930	7.159490585
320	320	0.987999916	7.604671001
352	352	0.996499956	8.219264030

number of simulation runs=2000

Table 23: k=0.9990, n=400

m	m0	confidence	expected-dt
32	29	0.999999940	4.486933231
64	61	0.999499917	5.256338457
96	93	0.999999940	5.757544041
128	126	0.999999940	6.154276848
160	158	0.999999940	6.501723766
192	190	0.999999940	6.826681137
224	223	0.999499917	7.147979736
256	255	0.999499977	7.483013630
288	287	0.999999940	7.853713989
320	320	0.996999919	8.299383163
352	352	0.999999940	8.913649559

number of simulation runs=2000

Table 24: k=0.9995, n=400

m	m0	confidence	expected-dt
32	29	0.999999940	5.180879116
64	61	0.999999940	5.950780392
96	93	0.999999940	6.451091766
128	126	0.999999940	6.848046780
160	158	0.999999940	7.195386887
192	190	0.999999940	7.520209789
224	223	0.999999940	7.841574669
256	255	0.999499917	8.176671982
288	287	0.999999940	8.548100471
320	320	0.999499977	8.993374825
352	352	0.999999940	9.607625008

number of simulation runs=2000

Table 25: true m=640, m0=606, n=4000, number of simulation runs=2000

mest	mest0	expected-dt	confidence	k(mest)
320	283	3.41542983	0.99699962	0.9939
640	606	3.44064760	0.99749953	0.9940
960	929	3.47614026	0.99899948	0.9942
1280	1253	3.50661421	0.99899954	0.9944
1600	1576	3.56079173	0.99899942	0.9947
1920	1899	3.63243365	0.99949944	0.9950
2240	2223	3.69857264	0.99949974	0.9953
2560	2546	3.83282208	0.99949956	0.9959
2880	2869	4.06088781	0.99949974	0.9968
3200	3192	4.57958984	0.99949974	0.9981
3520	3516	5.94781780	0.99949992	0.9995

Table 26: n=4000, mest equals m, number of simulation runs=2000

m	m0	confidence	expected-dt
320	283	0.997499466	2.632653475
640	606	0.997499526	3.440647602
960	929	0.997999489	3.980712652
1280	1253	0.997999668	4.409400940
1600	1576	0.997499764	4.811255932
1920	1899	0.998599596	5.208167103
2240	2223	0.998999715	5.595121861
2560	2546	0.997999668	6.063158512
2880	2869	0.997499883	6.659555435
3200	3192	0.997499883	7.619450092
3520	3516	0.997499943	9.593208313

§ 5.5 Comparisons

Based on the results of the numerical computations and simulations, we have the following conclusions. If the burn-in lot size is small, n=400, and the true value of m can be estimated reasonably accurately, then using Procedure 0 is the best choice. The reason is that Procedure 0 will give us the exact confidence with the minimum expected duration of burn-in if m is known. For example, let us look at the case m = mest = 64. Procedure 0 has t-delta = 4.34, Procedure I has E(D) = 5.2 and Procedure II has E(D) = 4.00. Here, the expected duration for Procedure II is less than t-delta of Procedure 0. This difference can be due to the randomness of the number strings generated by computer. In this example, the expected duration of burn-in for Procedure I is the longest one. Moreover, in this small lot size case, if the dispersion in m is large or the value of m is unpredictable, Procedure I would be a good choice, because Procedure I is less sensitive to the assumed m, mest, than Procedure 0. Here, we may use Procedure II, but the large sample theory for this procedure is not applicable and k must be obtained either from a direct calculation or simulation. The value of k obtained by using simulation turns out to give us the best burn-in procedure: it is not sensitive to the assumed value of m and its duration of burn-in is close to the minimum requirement.

On the other hand, if the burn-in lot size is large, n=4000, Procedure II is recommended. In this case, Procedure II is not only very insensitive to the assumed value of m but also its expected duration of burn-in is close to the minimum required duration of burn-in. Here, we may use Procedure 0, but the risk of not achieving our reliability goal is very great if the value of m is under-estimated. Procedure I is less sensitive than Procedure 0, but it requires a longer duration of burn-in.

CHAPTER 6

CONCLUSION

§6.1 Summary

Three burn-in procedures have been considered in this research. The performances of these procedures were compared in the previous chapter. When the lot size is small, we can consider using Procedure 0 if m can be accurately estimated, or using Procedure I. In this small lot size case, if a burn-in procedure which is very insensitive to the assumed value of m is required and the duration of burn-in is required to be as short as possible, then we can try to use simuation to find an appropriate k and use Procedure II. When the lot size is large, Procedure II is the first procedure to be considered. This procedure is very insensitive to the assumed value of m and its expected duration of burn-in is very close to the duration of burn-in when the number of defective items is known, i.e., the minimum required burn-in time. In this case, Procedure 0 is unable to achieve our reliability if m is under estimated and its expected duration of burn-in is too long if m is over estimated too much. Here, we can consider to use Procedure 0 only if we can very accurately estimate the unknown value m. For the large lot size case, Procedure I is not recommended. The reasons are: if we can accurately estimate the true value of m, then Procedure 0 performs better than Procedure I; and if we cannot accurately estimate the true value of m, then Procedure I is much more sensitive than Procedure II and it has longer expected duration of burn-in than Procedure II.

§6.2 Related Research and Future Research

The original motivation of this dissertation was to design non-replacement burn-in procedures for a batch of semiconductors with a known life time distribution (parameters are known) and an unknown proportion of defective items. The cost of burn-in is considered in this research, but no cost function is given here. Burn-in procedure with replacement is not considered, either. Burn-in with known life time distribution but with unknown parameters should be considered, too. In addition, the consideration of component burn-in and of system burn-in are not the same. Some of these topics are mentioned in Jensen and Petersen (1982)

For consideration of cost, see Kuo and Kuo (1983) for a very good summary about the existing burn-in cost models. A very comprehensive list of the existing papers about cost consideration is given in this paper. The references about this subject will not be given here. Kuo and Kuo also have a good discussion about cost minimization and savings. In addition, some warnings on cost modeling and cost optimization are presented in their paper.

A with-replacement burn-in procedure for the case that the parameter of the life time distribution of the defective items is unknown, was considered briefly during the preparation of this dissertation. If the life time of the perfect items is assumed infinity, then some process that is derived from the burn-in procedure is a birth-and-death process. Statistical inference about the parameters of a birth-and-death process can be found easily in Billingsley (1961) or Basawa and Prakasa Rao (1980).

Another without replacement procedure was investigated during the preparation of this dissertation. This procedure is obtained by finding an estimator of m . From Epstein and Sobel (1954), we know that 2 times the sum of the life times of the first j failed defective items plus 2(m-j) times the lifetime of the jth failed defective item has a χ^2 distribution with 2j degrees of freedom. Therefore, an estimator of m can be calculated if an appropriate value is assumed for this χ^2 random variable (e.g., its expectation). A new stopping rule can be developed by replacing the MLE of m with this estimator of m. This is the idea of this procedure.The properties of the statistic for this procedure have been studied briefly: this statistic is a linear combination of order statistics and its limit goes to a nonstationary Gaussian process with known mean and variance. Some stopping times based on this statistic have been looked at, but no satisfactory stopping time has been obtained yet. It will be studied further in the future.

Appendix

We assume that M has a prior binomial distribution, say binomial(n,r). The following two tables compare the relationship between the t*s, for Procedure I, derived by using Equations (2.3.5) and (2.9.3). Here, r is E(M/n), $r_{.99}$ is the 99-percentile of M/n obtained by using the normal approximation, t^*_{mg} is the t* obtained by using Equation (2.9.3), and t*(5,r) is the t* obtained by using (2.3.5) with m = r•n. This last value is given as a reference for the t* which would be used if M were a known constant.

A small n is used in Table 27. If r = .040 is used, then $r_{.99}$ = .06282, t^*_{mg} = 1.0419. In addition, t*(5,.06282) is 1.2370. In this table, we can see that t^*_{mg} increases as r increases and $t^*(5,r_{.99})$ is a step function in r. (The function $t^*(5,r_{.99})$ is a step function in r becasuse it is a function of m-m0 and m-m0 is a step function in m (or r)). The value t^*_{mg} gets close to the corresponding $t^*(5,r_{.99})$ as r increases before $t^*(5,r_{.99})$ reaches a jump and then gets close again. Most of the time $t^*(5,r_{.99})$ is greater than t^*_{mg} except before $t^*(5,r_{.99})$ has a jump. In addition, this table shows us that t^*_{mg} is very close $t^*(5,r_{.99})$ when it is greater than $t^*(5,r_{.99})$.

A large n is used in Table 28. If r = .18 is used, then $r_{.99}$ = .194153, t^*_{mg} = 0.18767, t*(5,.194153) = 0.192248. We can see that $t^*(5,r_{.99})$ is always greater than t^*_{mg} in this table. Furthermore, the difference between t^*_{mg} and $t^*(5,r_{.99})$ is less than .02 for each r.

Table 27: n=400, 1-exp(-t)=.99, r=0.99, a=0.99

r=m/n	$r_{.99}$	t^*_{mg}	$t^*(5,r)$	$t^*(5,r_{.99})$
.02000	.03631	1.02266	1.01144	1.23700
.03000	.04987	1.03219	1.23700	1.23700
.04000	.06282	1.04191	1.23700	1.23700
.05000	.07539	1.05183	1.23700	1.23700
.06000	.08766	1.06195	1.23700	1.23700
.07000	.09972	1.07229	1.23700	1.23700
.08000	.11160	1.08285	1.23700	1.23700
.09000	.12334	1.09364	1.23700	1.23700
.10000	.13495	1.10466	1.23700	1.23700
.11000	.14645	1.11592	1.23700	1.23700
.12000	.15785	1.12744	1.23700	1.23700
.13000	.16917	1.13921	1.23700	1.23700
.14000	.18042	1.15126	1.23700	1.23700
.15000	.19159	1.16358	1.23700	1.23700
.16000	.20270	1.17619	1.23700	1.23700
.17000	.21376	1.18910	1.23700	1.23700
.18000	.22475	1.20232	1.23700	1.23700
.19000	.23570	1.21586	1.23700	1.23700
.20000	.24660	1.22974	1.23700	1.23700
.21000	.25745	1.24396	1.23700	1.23700
.22000	.26825	1.25855	1.23700	1.60943
.23000	.27902	1.27350	1.23700	1.60943
.24000	.28975	1.28885	1.23700	1.60943
.25000	.30044	1.30460	1.23700	1.60943
.26000	.31110	1.32078	1.23700	1.60943
.27000	.32172	1.33739	1.60943	1.60943
.28000	.33230	1.35446	1.60943	1.60943
.29000	.34286	1.37201	1.60943	1.60943
.30000	.35338	1.39006	1.60943	1.60943
.31000	.36388	1.40862	1.60943	1.60943

Table 28: n=4000, 1-exp(-t)=.99, r=0.99, a=0.99

r=m/n	$r_{.99}$	t^*_{mg}	$t^*(5,r)$	$t^*(5,r_{.99})$
.04000	.047219	0.16346	0.15970	0.166094
.06000	.068749	0.16650	0.16255	0.166094
.08000	.089994	0.16967	0.16619	0.166915
.10000	.111052	0.17297	0.17002	0.173930
.12000	.131971	0.17641	0.17403	0.178152
.14000	.152783	0.18000	0.17826	0.182599
.16000	.173505	0.18375	0.17826	0.182291
.18000	.194153	0.18767	0.18271	0.192248
.20000	.214736	0.19178	0.18740	0.192248
.22000	.235261	0.19608	0.19236	0.197494
.24000	.255733	0.20059	0.19761	0.203056
.26000	.276159	0.20533	0.19761	0.208964
.28000	.296541	0.21031	0.20318	0.215253
.30000	.316882	0.21556	0.20909	0.221960

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